A GENERAL MIXTURE-TRUNCATION METHOD FOR GENERATING BIVARIATE RANDOM VARIABLES FROM UNIVARIATE GENERATORS

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THESIS

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FOR GENERATING BIVARIATE RANDOM
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by

Myong Ho Shin

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A simple mixture-truncation method is put forward here to generate bivariate distributions with any marginal distribution from univariate generators.

A General Mixture-Truncation Method for Generating Bivariate Random Variables from Univariate Generators

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ABSTRACT

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TABLE OF CONTENTS

I. INTRODUCTION

In many simulation applications, it is required to generate bivariate random variables which have identical marginal distribution.

For example, in reliability problems, the assumption that two components have independent exponential failure times is very often unrealistic, and much effort has gone into deriving bivariate exponential random variables to handle these situations [Gaver (1972), Olkin & Marshall (1967), Downton (1970)]. Now except in very specific physical situations it may be difficult to specify the complete bivariate distribution of life time of each component. However, it may be realistic to specify the marginal distributions and some measure of dependence (usually the correlation coefficient) between the life time of each component. In this kind of situation, we can use bivariate random vectors having given marginal distribution and dependence to solve the problem in simulation. To generate these vectors, there exists some previous works and existing methods, discussed in Section II. As seen in Section II, most previous work is specific to specified marginal distributions and uses inverse transformation methods as a basic concept.

An example is the recent work by Johsnon and Tenenbein (1979), to generate a bivariate random vector (X,Y) which has marginal distribution $F_1(x)$, $F_2(y)$ and correlation ρ, by a weighted linear combination method.

Define

$$X = F_1^{-1}(H_1(U))$$

$$Y = F_2^{-1}(H_2(V))$$

or

$$Y = F_2^{-1}(1 - H_2(V))$$

where H_1 and H_2 are the cumulative distribution functions (c.d.f.) of U and V respectively and

$$U = U'$$

$$V = cU' + (1-c)V'$$

where U', V' are i.i.d. random variables with probability density functino g( ). In this procedure, F_1^{-1}, F_2^{-1}, and c are specific to the marginal distribution and correlation desired. The functions F_1^{-1} and F_2^{-1} are difficult to compute in most cases and the weighting factor c is also difficult to calculate. Moreover most of the work in univariate random number generation has been aimed at avoiding having to calculate inverse cumulative distribution functions such as $F_1^{-1}(\cdot)$ and $F_2^{-1}(\cdot)$. These are the reasons why many proposed methods are specific to a specified marginal distribution.

Again special properties of certain random variables such as infinite divisibility have been exploited to give easily generated bivariate random variables, often though with limited ranges of dependency. One very clever scheme by Gaver (1972) to generate bivariate exponential random variables uses the fact that the sum of a geometrically distributed number of exponential random variables (Y) is exponentially distributed and that the minimum of this geometrically distributed number of independent logistic random variables (Z) is exponential. Clearly when Y is large, Z is small. This scheme is of course very specific to exponential marginal distributions and, via an exponential transformation, to uniform random variables. To avoid these kinds of limitations and to make the generation of bivariate random variables simpler and more automatic in simulations we develop here a scheme presented in Jacobs and Lewis (1977).

This scheme, the mixture-truncation method, is a very general tool which requires only that a method be available for generating random variables with the desired marginal distribution. The mixture-truncation method scheme for generating bivariate random variables is as follows.

Let $F(x)$ be the common marginal distribution, of the bivariate random variable (Y,Z). Let ρ be the desired correlation between Y and Z (which may or may not be attainable generally or in particular with the mixture-truncation scheme).

Define the transition matrix $\underline{P}$ as

$$\underline{P} = \begin{bmatrix} \alpha_1 & 1 - \alpha_1 \\ 1 - \alpha_2 & \alpha_2 \end{bmatrix}$$

with stationary vector

$$\underline{\pi}\,\underline{P} = \underline{\pi} = \left(\frac{1 - \alpha_2}{1 - \alpha_1 + 1 - \alpha_2}, \frac{1 - \alpha_1}{1 - \alpha_1 + 1 - \alpha_2}\right)$$

and let the range of the random variable X with distribution function F(x) be χ. Then generate (Y,Z) as follows.

1. (Initiallization)

   i) Choose an "allowable" x_o from (x_ℓ, x_u) (the allowable range of x_o will usually be smaller than χ and depends on ρ and F(x)).

   ii) Set $\pi_1 = F(x_o)$, $\pi_2 = 1 - \pi_1$ and compute α_1 and α_2.

   iii) Denote by X_1 the random variable X truncated to the left of x_o and by X_2 truncated to the right of x_o.

2. (Generate Y). Choose Y from X_1 with probability π_1 or choose Y from X_2 with probability π_2.

3. (Generate Z)

   i) If Y is chosen from X_1, choose Z from X_1 with probability α_1 or choose Z from X_2 with probability $1 - \alpha_1$.

ii) If Y is chosen from X_2, choose Z from X_1 with probability $1 - \alpha_2$ or choose Z from X_2 with probability α_2.

In Section III, we will show that the correlation ρ between Y and Z, if x_o is properly chosen, is

$$\rho = E[[\frac{Y - E\{Y\}}{\sigma_Y}] [\frac{Z - E\{Z\}}{\sigma_Z}]$$

$$= \beta M,$$

where

$$\beta = \alpha_1 - (1 - \alpha_2),$$

$$M = \frac{(\mu_1 - \mu_2)^2 \pi_1 \pi_2}{\sigma_X^2},$$

$$\mu_1 = E\{X_1\}, \quad \mu_2 = E\{X_2\},$$

$$\sigma_X^2 = VAR[X],$$

Moreover, the generated bivariate random vector (Y,Z) will have marginal distributions F(x) and correlation ρ and the joint distribution function of (Y,Z) will be

$$
F(y,z) = \begin{cases}
\frac{\alpha_1}{\pi_1} F(z)\, F(y) & \text{if} \quad z \leq x_o,\; y \leq x_o \\[2ex]
\{\alpha_1 + \frac{(1-\alpha_1)}{\pi_2}[F(z)-F(x_o)]\}F(y) & \text{if} \quad z > x_o,\; y \leq x_o \\[2ex]
\{\alpha_1 + \frac{(1-\alpha_1)}{\pi_2}[F(y)-F(x_o)]\}F(z) & \text{if} \quad z \leq x_o,\; y > x_o \\[2ex]
\alpha_1\pi_1 + (1-\alpha_1)[F(z)-F(x_o)]\, \frac{\pi_1}{\pi_2} & \text{if} \quad z < x_o,\; y > x_o \\[1ex]
\quad + \{(1-\alpha_2) + \frac{\alpha_2}{\pi_2}[F(z)-F(x_o)]\}[F(y)-F(x_o)] &
\end{cases}
$$

These relationships will be developed in Section III. Note that (Y,Z) may be continuous or discrete random variables or mixtures of both, though in this thesis we concentrate on continuous cases.

The key problem in this very simple algorithm comes at the initialization steps i) and ii). There are two degrees of freedom in the selection matrix $\underline{P}$ but setting $\pi_1 = F(x_o)$ constraints reduce to one degree of freedom. Specifying a desired correlation further constrains the degree of freedom, though not completely. Subject to the constraint that α_1 and α_2 are probabilities there may be

a. no values x_o which will give (Y,Z) with correlation ρ

b. one value x_o

c. a range $\bar{\chi}_o$.

The main numerical problem of initialization step (i) then is to compute the range of allowable x_o for a given ρ and $F(x)$.

The main statistical problem is then to choose which of the bivariate random vectors (Y,Z) indexed by $x_o \in \overline{\chi}_o$ to use. An alternate solution to the statistical problem, giving another algorithm is then to let x_o have some distribution in the range $\overline{\chi}_o$, possibly uniform or triangular, this not only alleviates the problem of picking a particular $x_o \in \overline{\chi}_o$ but it also smoothes out the distribution of (Y,Z) and possibly makes it continuous. The computation of $\overline{\chi}_o$ for given ρ is illustrated in subsequent sections for uniform, exponential and gamma marginals for (Y,Z).

Before doing this and developing, in Section III, the results already given here, we review a few existing methods for generating bivariate random variables in Section II. These are methods which seem fairly tractable, for use on a computer.

II. REVIEW OF EXISTING METHODS

There are several existing methods for generating bivariate random vectors, but most of them are specific to particular marginal distribution and use inverse transformation method. The inverse transformation method is a very useful univariate procedure which, unfortunately, is not possible to use with many distributions because it is difficult and/or uneconomic to compute the inverse functions. We survey here some of the methods which are germane to this thesis, in particular concentrating on bivariate random variables with uniform, exponential and gamma marginals. First we review the problem of determining the range of correlation coefficient ρ which can be obtained for bivariate distribution with given marginal distributions, again considering only the continuous case.

A. RANGE OF CORRELATION COEFFICIENT ρ

Suppose that Y, Z are random variables with an arbitrary joint distribution $F(y,z)$ with finite second moments. Then in general the correlation coefficient ρ can take all values in the closed interval $[-1,1]$. But with specific marginal distribution $F_1(y)$ and $F_2(z)$, the class of all $F(y,z)$ need not attain the values of -1, 1, of ρ. The necessary and sufficient conditions that there exist determination of $F(y,z)$ with ρ equal to 1 and -1 are given by Moran (1967) as follows.

i) there exist constants α and β such that $\alpha Y + \beta$ has the same distribution as Z,

ii) the distribution of Z is symmetrical about its mean.

To see this, rescale Y to have the same mean and variance as Z. If there exists an $F(y,z)$ such that $\rho = 1$ we have $E[Z - Y]^2 = 0$, so that $Y = Z$ with probability one. Then Y must have the same distribution as Z. On the other hand if there exists an $F(y,z)$ such that $\rho = -1$ we shall have $E[Z + Y]^2 = 0$, and $Y = -Z$ with probability one. Thus if both bounds are attainable Z must have a symmetric distribution.

Given general marginal distributions $F_1(y)$ and $F_2(z)$, Mardia (1970) showed Frechet bounds as

$$\max[0, F_1(y) + F_2(z) - 1] \leq F(y,z) \leq \min[F_1(y), F_2(z)] \qquad \text{(II-A-1)}$$

From this we can find the range of possible values of ρ. For simplicity we now confine our consideration to distributions $F(y,z)$ of positive random variables whose derivatives $F_1'(y)$ and $F_2'(z)$ are strictly positive for $y > 0$, $z > 0$, respectively. Suppose also that the variates are scaled to have unit variances. Let $G_1(u)$, $G_2(v)$ be the inverse functions of $F_1(y)$, $F_2(z)$, i.e.

$$F_i[G_i(w)] = w$$

where $0 \leq w \leq 1$, $i = 1, 2$. Then the correlation coefficient

between Y and Z is given by the equation

$$\rho = \int_0^\infty \int_0^\infty y\, z\, dF(y,z) - E[Y]\, E[Z] \qquad \text{(II-A-2a)}$$

$$= \int_0^1 \int_0^1 G_1(u)\, G_2(v)\, dK(u,v) - E[Y]\, E[Z] \qquad \text{(II-A-2b)}$$

where $K(u,v)$ is the joint distribution of the quantities $U = F_1(y)$, $V = F_2(z)$. Then U,V are jointly distributed on the square $0 \leq U \leq 1$, $0 \leq V \leq 1$, in such a way that the marginal distributions are uniform on the unit intervals. From expression (II-A-1) the minimum correlation is attained when the probability is concentrated uniformly on the line $U + V = 1$. The minimum value of ρ then is

$$\rho_{min} = \int_0^1 G_1(u)\, G_2(1-u)\, du - E[Z]\, E[Y] \qquad \text{(II-A-3)}$$

The corresponding $F(y,z)$ will be a singular distribution with all the probability concentrated on the line $F_1(y) + F_2(z) = 1$. In fact Y and Z are an antithetic pair. The maximum value of ρ is attained when the probability is concentrated uniformly on the line $U = V$. Then the maximum value of ρ is

$$\rho_{max} = \int_0^1 G_1(u)\, G_2(u)\, du - E[Z]\, E[Y], \qquad \text{(II-A-4)}$$

with the corresponding probability concentrated on the line $F_1(y) = F_2(z)$.

By using this result we will get lower and upper bounds of the correlation coefficient for uniform, exponential and gamma marginal cases. In the uniform marginal distribution case, we can see $\rho_{min} = -1$ and $\rho_{max} = 1$ from Moran's condition ii), i.e., uniform distribution is symmetric about its mean. For the exponential marginal distribution case suppose Y and Z have exponential distributions with unit mean and variance. Then

$$F_1(y) = 1 - e^{-y}, \quad F_2(z) = 1 - e^{-z}$$

and the inverse functions are

$$G_1(u) = -\ln(1-u),$$

$$G_2(v) = -\ln(1-v).$$

From equations (II-A-3) and (II-A-4)

$$\rho_{min} = \int_0^1 G_1(x)\, G_2(1-x)\, dx - E[Y]\, E[Z]$$

$$= \int_0^1 \ln x \ln(1-x)\, dx - 1 = 1 - \frac{\pi^2}{6}$$

$$\simeq -0.64493$$

$$\rho_{max} = \int_0^1 G_1(x)\ G_2(x)\ dx - E[Y]\ E[Z]$$

$$= \int_0^1 \ln x \ln x\ dx - 1 = 1$$

The ρ_{min} can be attained when all Y and Z are concentrated on the line $e^{-Y} + e^{-Z} = 1$. Also the ρ_{max} can be attained when all Y and Z are concentrated on the line $e^{-Y} = e^{-Z}$, i.e., Y = Z.

For the gamma marginal distribution case, suppose that Y and Z have gamma type distributions with probability density functions

$$f_1(y) = \frac{1}{\Gamma(\alpha)} e^{-Y} z^{\alpha-1} \qquad \alpha > 0 \quad , y \geq 0$$

$$f_2(z) = \frac{1}{\Gamma(\beta)} e^{-z} z^{\beta-1} \qquad \beta > 0 \quad , z \geq 0 .$$

Then

$$E[Y] = \alpha, \quad E[Z] = \beta, \quad VAR[Y] = \alpha \quad VAR[Z] = \beta .$$

The ρ_{min} will be attained when the probability density is concentrated on the line

$$\frac{1}{\Gamma(\alpha)} \int_0^Y e^{-x}\ x^{\alpha-1}\ dx + \frac{1}{\Gamma(\beta)} \int_0^z e^{-x}\ x^{\beta-1}\ dx = 1 .$$

This defines y uniquely as a function of z which can be written $y = A(z)$. The ρ_{min} is then, on rescaling the covariance

$$\rho_{min} = \{ \int_0^\infty u\, A(u)\, f_1(u)\, du - \alpha\beta\}/(\alpha\beta)^{1/2}$$

When α, β become large, ρ_{min} tends to -1. And the ρ_{max} will be attained when the probability density is concentrated on the line

$$\frac{1}{\Gamma(\alpha)} \int_0^y e^{-x}\, x^{\alpha-1}\, dx = \frac{1}{\Gamma(\beta)} \int_0^z e^{-x}\, x^{\alpha-1}\, dx$$

This defines y as a function of z which can be written $y = B(z)$. The ρ_{max} then is, on rescaling the covariance

$$\rho_{max} = \{ \int_0^\infty u\, B(u)\, f_1(u)\, du - \alpha\beta\}/(\alpha\beta)^{1/2}$$

when $\alpha = \beta$, ρ_{max} tends to 1. Schmeiser and Ram Lal (1979) showed the obtainable correlations between random variables Y and Z having gamma marginal distribution with density function

$$f_i(x) = [(x/\beta_i)^{\alpha_i - 1} \exp(-x/\beta_i)/(\beta_i \Gamma(\alpha_i))]$$

for $x > 0$, $\alpha_i > 0$, $\beta_i > 0$, $i = 1, 2$.

The Figures (II-a) and (II-b) show the obtainable correlation as a function of α_2, given $\alpha_1 = 1$ and 5, for $\beta_i = 1$.

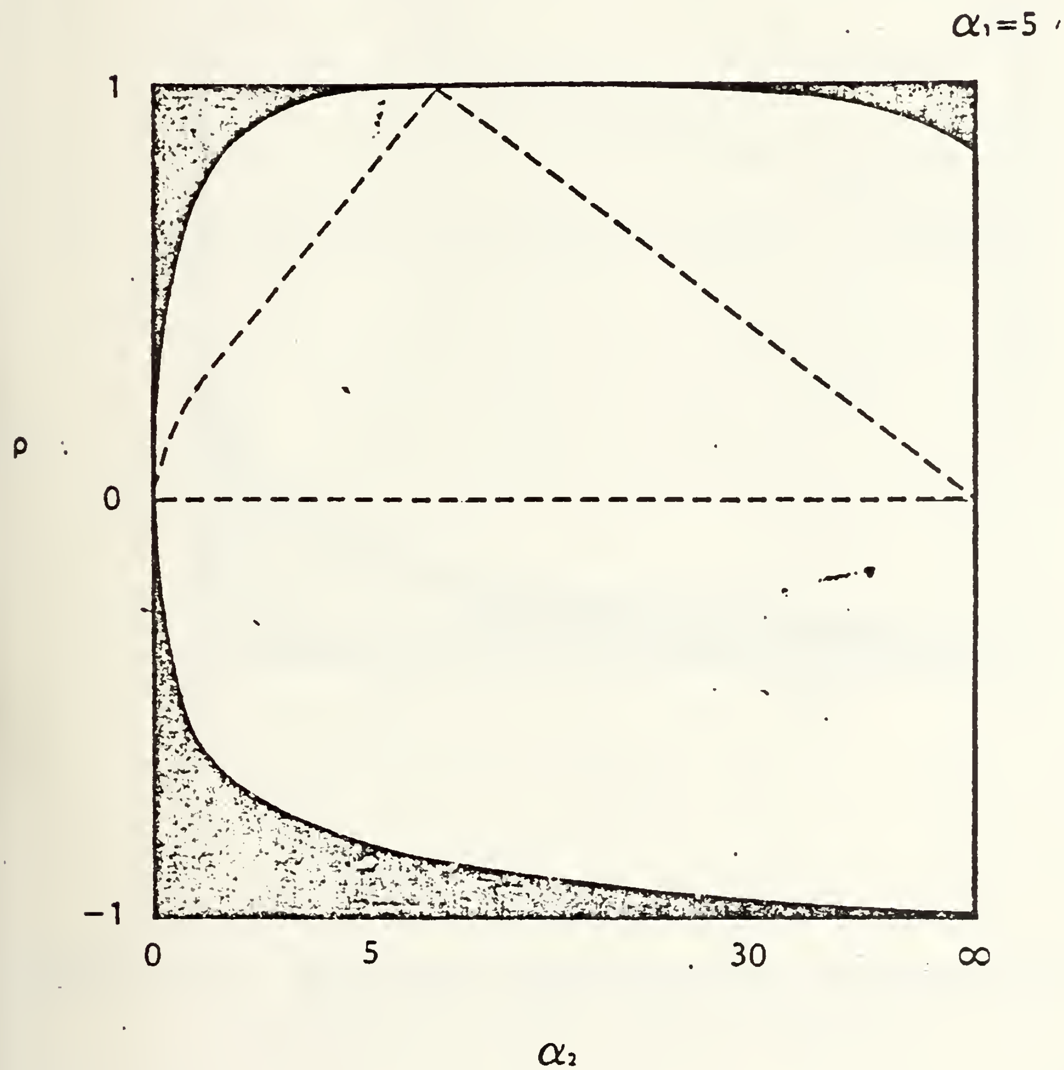


Figure II-a. Obtainable correlation as a function of α_2 for $\alpha_1 = 1$, $\beta_1 = 1$, by Schmeiser

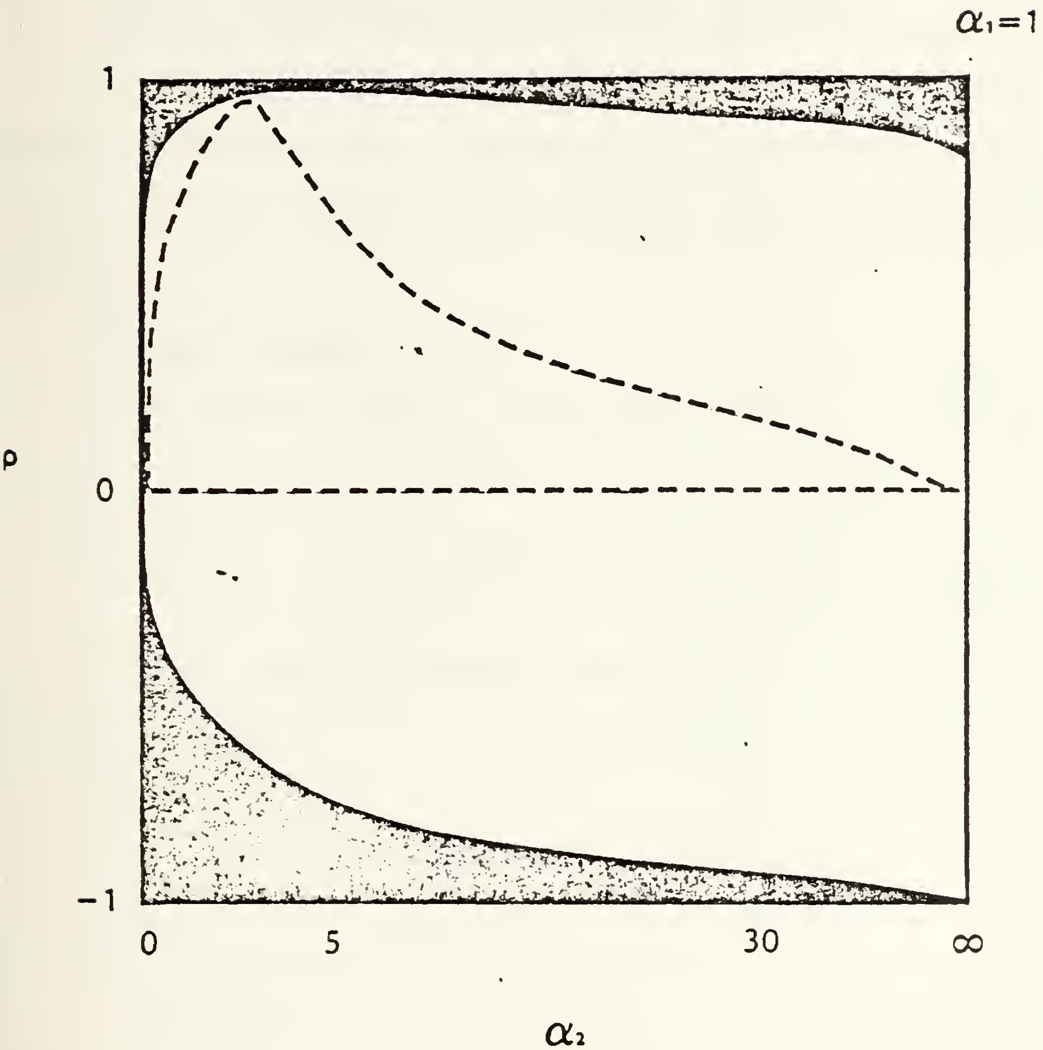


Figure II-b. Obtainable correlation as a function of α_2 for $\alpha_1 = 5$, $\beta_1 = 1$, by Schmeiser

B. GENERAL METHOD REVIEW

1. Johnson and Tenenbein's General Method

A general method of constructing a bivariate distribution, whose marginal distribution functions are $F_1(x)$ and $F_2(y)$, is proposed by Nataf (1962), can be represented as follows.

General Method

i) Consider any two continuous random variables U and V with probability density functin h(u,v).

ii) Let $X' = H_1(u)$ and $Y' = H_2(v)$, where $H_1(u)$ and $H_2(v)$ are the cumulative distribution functions of U and V, respectively.

iii) Define

$$X = F_1^{-1}(X') = F_1^{-1}[H_1(u)] \qquad \text{(II-B-1)}$$

and

$$Y = F_2^{-1}(Y') = F_2^{-1}[H_2(v)] \qquad \text{(II-B-2a)}$$

or

$$Y = F_2^{-1}(1 - Y') = F_2^{-1}[1 - H_2(v)] \qquad \text{(II-B-2b)}$$

Since X', Y' and 1-Y' are uniformly distributed over the interval [0,1], X defined by expression (II-B-1) and Y defined either by (II-B-2a) or (II-B-2b) will have a joint distribution whose marginal distribution functions are $F_1(x)$ and $F_2(y)$. The method is probably what one would think of first but again involves inverses. Based on this general method, Johnson and Tenenbein (1979) developed two procedures for generating (and also constructing) bivariate distributions whose marginal distributions and measures of dependence, as given by Kendall's τ or the grade correlation coefficient ρ_s, are specified. Note that these Kendall's τ and grade correlation coefficient ρ_s are not the same as the ordinary product moment correlation coefficient ρ we have been considering; these measures are discussed by Kendall (1962) and Kruskal (1958) and can be defined as follows. Let X and Y be continuous random variables having some joint probability density function. Let (X_1,Y_1), (X_2,Y_2) and (X_3,Y_3) be three independent pairs of observations having the same joint density function, then

$$\tau = 2\, P[(X_1-X_2)(Y_1-Y_2) > 0] - 1$$

$$\rho_s = 6\, P[(X_1-X_2)(Y_1-Y_3) > 0] - 3$$

The first procedure for generating bivariate pairs, called the WLC (weighted linear combination), defines

$$U = U' \qquad \text{(II-B-3)}$$

and

$$V = cU' + (1-c)V' \qquad \text{(II-B-4)}$$

where $0 \leq c \leq 1$, U' and V' are independent and identically distributed random variables with probability density function $g(\cdot)$. Then X and Y are obtained from the general method using equation (II-B-1) and (II-B-2a) if the dependence measure is positive or equations (II-B-1) and (II-B-2b) if the dependence measure is negative.

In order to apply the WLC procedure, we must obtain expressions for $H_1(u)$, $H_2(v)$, $\sigma_s(u,v)$ and $T(u,v)$, in terms of c and $g(\cdot)$. The values of $H_1(u)$ and $H_2(v)$ allow us to apply the general method for a given choice of c and $g(t)$. The expressions for $T(u,v)$ and $\sigma_s(u,v)$ allow us to specify c for a given choice of $g(\cdot)$ in terms of the required value of either T or σ_s. From equations (II-B-3) and (II-B-4)

$$H_1(u) = \int_{-\infty}^{u} g(t)\, dt \qquad \text{(II-B-5)}$$

$$H_2(v) = \iint_{R_1} g(u')\, g(v')\, du'dv' \qquad \text{(II-B-6)}$$

where

$$R_1 = \{(u',v'):\ cu' + (1-c)v' \leq v\}$$

and the joint density function of u and v is

$$h(u,v) = \frac{1}{1-c}\, g(u)\ g\left(\frac{v-cu}{1-c}\right) \qquad \text{(II-B-7)}$$

$$\rho_s(u,v) = 12 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_1(u)H_2(v)h(u,v)\,du\,dv - 3 \qquad \text{(II-B-8)}$$

$$\tau(u,v) = 4 \int_{-\infty}^{0} G_2\left(\frac{1-c}{c}\ t\right)\ g_2(t)\ dt\ , \qquad \text{(II-B-9)}$$

where

$$g_2(t) = \int_{-\infty}^{\infty} g(w+t)\ g(w)\ dw$$

$$G_2(t) = \int_{-\infty}^{\infty} g_2(x)\ dx$$

Using equations (II-B-5), (II-B-6), (II-B-7), (II-B-8) and (II-B-9), we have to evaluate $H_1(u)$, $H_2(v)$, $h(u,v)$, $\tau(u,v)$, and $\rho_s(u.v)$ for all values of c and for specified $g(\cdot)$. As Johnson and Tenenbein mentioned in their report, these integrals are quite tedious to perform, so they gave some computation results in their report.

The second procedure, called TVR (trivariate reduction) is discussed by Mardia (1970). In this case U and V are defined as

$$U = U' + \beta Z' \quad \text{(II-B-10)}$$

$$V = V' + \beta Z' \quad \text{(II-B-11)}$$

for $0 \leq \beta < \infty$, and U', V', and Z' are independent and identically distributed random variables with probability density function $g(\cdot)$.

Like the WLC procedure, one needs to define $H_1(u)$, $H_2(v)$, $h(u,v)$, $\rho_s(u,v)$ and $T(u,v)$ as a function of β and $g(\cdot)$. From equations (II-B-10) and (II-B-11) it follows that

$$H_1(u) = H_2(v) = \int\int_{R_3} g(u')\, g(z')\, du'\, dz' \quad \text{(II-B-12)}$$

where

$$R_3 = \{(u',z'):\ u' + \beta z' \leq u\},$$

$$h(u,v) = \int_{-\infty}^{\infty} g(u-\beta z) g(v-\beta z)\, g(z)\, dz \quad \text{(II-B-13)}$$

and

$$T(u,v) = 4 \int_{-\infty}^{\infty} [G_2(\beta t)]^2 \, g_2(t) \, dt - 1 \qquad \text{(II-B-14)}$$

where

$$g_2(t) = \int_{-\infty}^{\infty} g(w+t) \, g(w) \, dw$$

$$G_2(t) = \int_{-\infty}^{\infty} g_2(x) \, dx$$

using equations (II-B-12), (II-B-13), (II-B-14) and (II-B-8) the same computations are needed.

In this general WLC and TVR methods, there are two problems. One is the need for inverse transformations. The other is the need to generate coupled random variables to create dependence after the inverse transformation is applied. Doing this by linear combinations is not necessarily the simplest and most felicitous with respect to calculation of correlations and range of correlations.

A significant simplification can be achieved by obtaining a "smooth" bivariate uniform pair (U,V) whose correlation can be varied between the limit of correlation which can be obtained for uniform marginals. These limits are (-1,1) since the antithetic pair (U,1-U) has correlation -1.

Note that the problem is essentially one of obtaining a bivariate uniform random variable with positive correlation because

$$\operatorname{corr}(U,V) = \rho_{u,v} = \begin{cases} -\rho_{U,1-V} = \operatorname{corr}(U,1-V) \\ -\rho_{1-U,V} = \operatorname{corr}(1-U,V) \end{cases}$$

Two fairly simple schemes for obtaining bivariate uniform distributions are discussed in the next section.

C. REVIEW OF SOME SPECIFIC METHODS

The mixture-truncation method is a general algorithm for generating bivariate pairs (Y,Z) with given marginal distributions and requires only the availability of a generator for the marginal distribution and calculation of constants. Unless randomization is used, however, it does give a bivariate distribution with a discontinuous joint distribution, as in Section (III-D). This could be a disadvantage. It is interesting therefore to compare it to other generations which are specific to the given marginal distribution and we do this here for the cases for which the mixture-truncation method is illustrated in this thesis, namely uniform, exponential and gamma.

There are many schemes available for uniform and exponential cases though it should be noted that smooth, simply generated pairs with easily calculated correlations have only recently been available. In particular we give here two

very recent schemes for smooth bivariate uniforms which are easily generated and whose correlations can be calculated; similarly two schemes for exponential which are essentially the uniform schemes after transformation. When one comes to gamma marginals one is again in difficulty and we describe a recent scheme by Schmeiser and Ram Lal which involves however, computation of the inverse gamma distribution function and very difficult initialization computations. The mixture-truncation method is quite competitive here. In fact if one can compute the inverse gamma distribution then bivariate gamma's can be computed as $[Y = F^{-1}(U),\ Z = F^{-1}(V)]$ where U and V are a bivariate uniform generated by the schemes mentioned above; this would be simpler for any marginal distributions than some of the general schemes mentioned in the previous sections. Comparisons of the generating schemes given in this section with the mixture-truncation method are given in Section IV-D and Section V-D.

1. Lawrance and Lewis's Bivariate Uniform

Lawrance and Lewis (1979) show that a simple transformation of the NEAR(1) (New Exponential Autoregressive) process gives a two-parameter family of Markovian random variables with uniform marginal distributions. This method generates a correlated uniform pair as a multiplicative mixture of uniform variables, where the parameters α and β take on values between zero and one, with the condition that they not both be one. Let Y be a uniform [0,1] random variable, and define

$$Z = \begin{cases} \varepsilon\, Y^{\beta} & \text{wp } \alpha \\ \varepsilon & \text{wp } 1-\alpha \end{cases}$$

where

$$\varepsilon = \begin{cases} U & \text{wp } \dfrac{1-\beta}{1-(1-\alpha)\beta} \\ U^{(1-\alpha)\beta} & \text{wp } \dfrac{\alpha\beta}{1-(1-\alpha)\beta} \end{cases}$$

where U is an independent identically distributed uniform (0,1) random variable. The correlation structure is defined as follows as a function of α and β;

$$\rho_{Y,Z} = \frac{3}{2+\beta}\left(\frac{\alpha\beta}{1+(1-\alpha)\beta}\right)$$

The model can be reduced to a one-parameter model by suitably relating α and β, e.g., $\alpha = \beta$ and all positive values of ρ can be obtained. Consequently Y and 1-Z will have a full range of negative correlations. A generating procedure for this bivariate pair of uniform random variables is as follows.

<u>Generating Procedure</u>

1. (Initialization). For given correlation, find suitable parameter values α and β.
2. Generate U_1, set $Y \leftarrow U_1$; $r = 1-\alpha$, $P \leftarrow \dfrac{1-\beta}{1-(1-\alpha)\beta}$.
3. Generate U_2.

4. If $U_2 \le r$, set $U \leftarrow \frac{U_2}{r}$ and go to 8.

5. Otherwise $U \leftarrow \frac{U_2 - r}{1 - r}$.

6. If $U \le P$, set $Z \leftarrow Y^{\beta} \frac{U}{1-P}$ and go to 10.

7. Otherwise, set $Z \leftarrow Y^{\beta} (\frac{U - P}{1 - P})^{r\beta}$ and go to 10.

8. If $U \le P$, set $Z \leftarrow \frac{U}{P}$ and go to 10.

9. Otherwise, set $Z \leftarrow (\frac{U - P}{1 - P})^{r\beta}$.

10. Go to 2 and repeat until the desired number of bivariate uniform variables (Y,Z) are obtained.

The program is listed in the appendix and the scatter plot and bivariate histogram of resulting output are in Section IV-D.

2. <u>Bivariate Uniform By Transformation of Gaver's Bivariate Exponential</u>

In Gaver's bivariate exponential to be discussed in Section II-C-4, we can define a bivariate exponential random vector, (Y,Z) with correlation $\rho = - P/2$ as follows:

$$Y = \ln[\frac{1}{U^{1/N}(1-P)} - \frac{P}{1-P}] \qquad \text{(II-C-1)}$$

and Z is, conditional upon N = n, a gamma random variable with shape parameter n and mean n(1-p), where N is a geometric random variables with probability density function $f(x) = p(1-p)^{x-1}$, x = 1, 2, ... and U is a uniform random variable over [0,1]. From this we can generate a pair of

uniform random variables (V,W) by transformation. Since we know that to generate an exponential random variable X from a uniform random variable U[0,1), we use the following transformation:

$$X = - \ln(1 - U) \qquad \text{(II-C-2)}$$

The resulting X has the exponential distribution with unit mean. From equations (II-C-1) and (II-C-2), we can generate V, a uniform random variable over [0,1]:

$$V = \frac{(1-p)\ U^{1/N}}{1 - P\ U^{1/N}}$$

Further also we can generate W, a uniform random variable over [0,1]:

$$W = \left(\prod_{i=1}^{N} U_i \right)^{1-P}$$

This follows since we know that Z has, conditionally upon N = n, the gamma distribution with shape parameter n and mean is n(1-p).

In general the resulting V and W will be negatively correlated and (V,1-W) will be a positively correlated pair. Because the correlation structure will be changed by transformation, the resulting (V,W) need not have the same

correlation with (Y,Z). The correlation coefficient of (V,W) will be computed as a function of P as follows.

$$\rho = 12\, E[VW] - 3$$

where

$$E[VW] = E_N[E[VW|N=n]]$$

$$= E_N\left[\int_0^1 \frac{(1-p)U^{1/n}}{1-P\,U^{1/n}}\, du - \left[\int_0^1 u^{1-P}\, du\right]^n \right]$$

Unfortunately this computation of ρ as a function of P is difficult. As an example we used here the same function for ρ as holds in the exponential case.

<u>Generating Procedure</u>

1. (Initialization)

   i) Compute P for given correlation ρ

   ii) Choose N from a geometric distribution with parameter 1-P

2. Generate a uniform [0,1] random variable U_1 and define

$$V = \frac{(1-P)\, U_1^{1/n}}{1 - P\, U_1^{1/n}}$$

3. Generate N = n uniform [0,1] random variables U_i, i = 1, ..., n, and define

$$W = \left(\prod_{i=1}^{n} U_i \right)^{1-p}$$

4. Deliver (V,W) and go to 2 until a sufficient number of bivariate pairs have been generated.

The program is listed in the appendix and the scatter plot and bivariate histogram of resulting random vectors are in Section IV-D.

3. Marshall and Olkin's Positively Correlated Bivariate Exponential

Suppose X is the age, or length of service of the first device at the time of death, governed by two independent Poisson processes $Z_1(t)$, $Z_{12}(t)$ with parameters λ_1, λ_{12} respectively and Y the same of the second device, governed by two independent Poisson processes $Z_2(t)$, $Z_{12}(t)$ with parameters λ_2, λ_{12}, respectively. Further suppose the second device is placed in operating after a time δ later than the first device. Then the joint distribution of (X,Y) is defined as

$$P[X > x, Y > y] \equiv F(x,y)$$

$$= \begin{cases} \exp\{-\lambda_1 x - \lambda_2 y - \lambda_{12}\max[x, y+\min(x,\delta)]\} & \text{if } \delta \geq 0 \\ \exp\{-\lambda_1 x - \lambda_2 y - \lambda_{12}\max[y, x+\min(y,\delta)]\} & \text{if } \delta \leq 0 \end{cases} \tag{II-C-3}$$

If $\delta = 0$, then equation (II-C-3) reduces to

$$P[X > x, Y > y] \equiv \bar{F}(x,y)$$

$$= \exp\{-\lambda_1 x - \lambda_2 y - \lambda_{12} \max(x,y)\}$$

Note that since the marginal distributions do not depend on δ,

$$E[X] = \frac{1}{\lambda_1 + \lambda_{12}},$$

$$VAR[X] = \frac{1}{(\lambda_1 + \lambda_{12})^2},$$

$$E[Y] = \frac{1}{\lambda_2 + \lambda_{12}},$$

$$VAR[Y] = \frac{1}{(\lambda_2 + \lambda_{12})^2}$$

We also have, for $\delta \geq 0$

$$E[XY] = \frac{1}{(\lambda_1 + \lambda_{12})(\lambda_2 + \lambda_{12})} + \frac{\lambda_{12}\, e^{-(\lambda_1+\lambda_{12})}}{(\lambda_1 + \lambda_{12})(\lambda_2 + \lambda_{12})}$$

and the correlation

$$\operatorname{corr}(X,Y) = \rho_{x,y} = \frac{\lambda_{12}}{\lambda}\, e^{-(\lambda_1+\lambda_{12})\delta} \qquad \text{(II-C-4a)}$$

where

$$\lambda = \lambda_1 + \lambda_2 + \lambda_{12}.$$

If $\delta = 0$, then the equation (II-C-4a) reduces to

$$\rho_{xy} = \frac{\lambda_{12}}{\lambda} \qquad \text{(II-C-4b)}$$

This characterization can be represented in terms of independent random variables. For $\delta \geq 0$, if there exist independent exponential random variables U_1, U_2, U_3 and U_4 with respective parameters λ_1, λ_2, λ_{12}, λ_{12}, then

$$X = \min(U_1, U_3)$$

$$Y = \begin{cases} \min(U_2, U_3-\delta) & \text{if} \quad U_3 \geq \delta \\ \min(U_2, U_4) & \text{if} \quad U_3 < \delta \end{cases}$$

It can be verified formally from the relation

$$P[X > x, Y > y] = P[U_1 > x, U_2 > y]\{P[U_3 > x, U_3 > y+\delta \mid U_3 > \delta]$$

$$\times P[U_3 > \delta]\} + P[U_4 > y]P[U_3 > x \mid U_3 < \delta]P[U_3 > \delta]\}$$

In the $\delta = 0$ case, the representation yields

$$X = \min(U_1, U_3)$$

$$Y = \min(U_2, U_3)$$

This bivariate distribution has a line-discontinuity, if $E[X] = E[Y]$, along the line $x = y$, i.e., in the case where $x = y = U_3$. Thus this bivariate exponential is not as smooth as others. In particular the NEAR(1) process of Lawrance and Lewis generates bivariate exponentials with a continuous density function. But if we consider the case $\delta \geq 0$, the line-discontinuity will be removed. In either case, the resulting (X,Y) pairs always have positive correlation as shown in equations (II-C-4a) and (II-C-4b). If we use these methods to generate positively correlated random vectors (X,Y) for exponential marginal distribution with unit mean and given ρ, we have to compute λ_1, λ_2 and λ_{12} for given correlations to generate independent exponential random variables. For simplicity we will consider the method with $\delta = 0$. From the relationship $\lambda = \lambda_1 + \lambda_2 + \lambda_{12}$,

$$E[X] = \frac{1}{\lambda_1 + \lambda_{12}} = 1$$

$$E[Y] = \frac{1}{\lambda_2 + \lambda_{12}} = 1$$

$$\rho = \frac{\lambda_{12}}{\lambda} ,$$

we can compute λ_1, λ_2 and λ_{12} as functions of ρ.

$$\lambda_{12} = 2\rho/(1 + \rho)$$

$$\lambda_1 = \lambda_2 = 1 - \lambda_{12} \cdot$$

Generating Procedure

1. (Initialization). Compute λ_1, λ_2 and λ_{12} for given $0 \leq \rho \leq 1$.

2. Generate three independent exponential random variables E_1, E_2 and E_3 with unit mean.

3. Rescale the independent random variables and set as

$$U_1 \leftarrow E_1/\lambda_1$$

$$U_2 \leftarrow E_2/\lambda_2$$

$$U_3 \leftarrow E_3/\lambda_{12}$$

4. Define X and Y as

$$X = \min[U_1, U_3]$$

$$Y = \min[U_2, U_3]$$

5. Deliver (X,Y) and go to 2 until a sufficient number of bivariate pairs have been generated.

The program is listed in the appendix and the scatter plot and bivariate histogram of resulting random vectors are shown in Section V-D.

4. Gaver's Negatively Correlated Bivariate Exponential

This negatively correlated bivariate exponential is generated from the following considerations. Suppose a particular system has N defective elements; here we assume N has geometric distribution with probability density function

$$f(x) = p(1-p)^{x-1}, \quad x = 1, 2, \ldots .$$

The time to failure of each defective element is T_i with density function $F(t)$. Further suppose repair is carried out after all defective elements are discovered. If we define $\underline{R}$ as repairing time which is distributed as $G^n(x)$, provided $N = n$, and define $\underline{T}$ as minimum of T_i, then the joint distribution function of $\underline{T}$ and $\underline{R}$ is obtainable from

$$P[\underline{T} > t, \underline{R} \leq x] = \sum_{n=1}^{\infty} (1 - F(t))^n \, P_n \, G^n(x)$$

where

$$p_n = (1-p)\, p^{n-1}, \quad 0 < p < 1, \quad n = 1, 2, \ldots$$

Then

$$E[\underline{R} | \underline{T} > t] = \frac{E[R]}{1 - P[1 - F(t)]}$$

where E[R] is the expected value of an element repair time. Moreover,

$$E[\underline{R}|\underline{T} = t] = E[R]\,\frac{1 + P[1 - F(t)]}{1 - P[1 - F(t)]}$$

$$= E[R]\;[1 + \frac{2P}{1 - P}(1 - \phi(t))]$$

where

$$\phi(t) = P[\underline{T} \leq t] = \frac{F(t)}{1 - P[1 - F(t)]}$$

From this regression, we know that if $\underline{T}$ is long, $\underline{R}$ is short and vice versa; the negative relationship is clearly present. Further we can calculate Cov[$\underline{R}$,$\underline{T}$) as

$$\mathrm{Cov}[\underline{R},\underline{T}) = P\,E[R]\,E[T](1 - \frac{E[T(1)]}{E[T]}) < 0,$$

where T(1) has the distribution of the smallest of a sample of two from $\phi(t)$.

It will not be shown that one can select $F(t) = 1 - \overline{F}(t)$ in such a way as to induce exponentially distributed time to system failure, $\underline{T}$. Since we know that exponential G, geometrically compounded, yields exponential $\underline{R}$, the outcome will be a ($\underline{T}$,$\underline{R}$) pair with exponential marginals and negative correlation.

If

$$F(t) = 1 - \overline{F}(t)$$

satisfies

$$\sum_{n=1}^{\infty} [\overline{F}(t)]^n \, p^{n-1} \, (1-p) = e^{-\lambda t}$$

then $\underline{T}$ is exponential with mean $1/\lambda$. Solution for F yields,

$$\overline{F}(t) = \frac{1}{p + (1-p)\, e^{\lambda t}} \qquad t \geq 0$$

which is a logistic distribution, left truncated at $t = 0$. If $\lambda = 1$, then $\underline{T}$ is exponential with unit mean, while $E[\underline{T}_{(1)}] = 1/2$, so

$$\text{corr}(\underline{R},\underline{T}) = -\frac{p}{2}\, E[\underline{R}]\; E[\underline{T}] \;.$$

If G is chosen so as to make $\underline{R}$ exponential then it may be shown that

$$\text{corr}(\underline{R},\underline{T}) = -\frac{p}{2} = -\frac{1}{2}[1 - \frac{1}{E[N]}]$$

Consequently, a greatest lower bound for the correlation in this model is $-\frac{1}{2}$.

Generating Procedure

1. If $0 < \rho < -0.5$, Set $P \leftarrow -2\rho$

2. Generate a geometric random variable N with parameter 1-P

3. Generate a uniform (0,1) random variable U and define Y as

$$Y = \ln\left[\frac{1}{U^{1/n}(1-p)} - \frac{p}{1-p}\right]$$

4. Generate a gamma random variable, Z, with shape parameter n, and mean is n(1-p)

5. Deliver (Y,Z) and go to 2 until a sufficient number of pairs are obtained.

For obtaining negatively correlated bivariate exponential random vectors, this algorithm is very simple and one of the few available. Moreover the correlation is known explicitly. The program is listed in the appendix and scatter plot and bivariate histogram of resulting random vectors are shown in Section V-D.

5. Arnold's Vibariate Gamma Generator

Arnold (1967) developed a Trivariate reduction method to generate bivariate gamma random vectors having positive correlation with

$$\text{corr } \rho < \min(\alpha_1, \alpha_2)/(\alpha_1\alpha_2)^{1/2} ,$$

where α_1 and α_2 are the given shape parameters for the marginal distributions. Letting "gamma (α,β)" denote the gamma distribution with shape parameter α and scale parameter β, so that

$$f(x) = \frac{1}{\beta\Gamma(\alpha)} \left(\frac{x}{\beta}\right)^{\alpha-1} \exp(-x/\beta),$$

where $\Gamma(\alpha)$ is the gamma function and

$$E[x] = \alpha\beta \quad , \quad VAR[x] = \alpha\beta^2,$$

the trivariate reduction algorithm proceeds as follows for given shape parameter values α_1, α_2 and ρ to generate unit scale gamma variate i.e., $\beta = 1$.

Generating Procedure

1. Generate $G_1 \sim$ gamma $(\alpha_1 - \rho(\alpha_1\alpha_2)^{1/2}, 1)$
2. Generate $G_2 \sim$ gamma $(\alpha_2 - \rho(\alpha_1\alpha_2)^{1/2}, 1)$
3. Generate $G_3 \sim$ gamma $(\rho(\alpha_1\alpha_2)^{1/2}, 1)$
4. Define Y and Z

$$Y \leftarrow G_1 + G_3$$

$$Z \leftarrow G_2 + G_3$$

These Y and Z are unit gamma variates with $\beta = 1$, but Y and Z can be multiplied by β_1 and β_2, respectively, to obtain any desired scaling. In the algorithm, G_3 is a common component of both Y and Z, thus inducing positive correlation.

When a positive, but not extreme, correlation is needed, trivariate reduction is an excellent algorithm, using univariate gamma generators. If the marginal distributions have the same shape parameter value α, then the correlation limitation is $0 \le \rho < 1$. But in the other case the upper limit is smaller than 1.

Note that this algorithm exploits the infinite divisibility of the marginal gamma variables and is applicable to any infinitely divisible marginal. It is commonly used, for instance, to get bivariate Poisson random variables.

6. Schmeiser's Bivariate Gamma Generator

Schmeiser (1979) developed a family of algorithms, any member of which can generate bivariate gamma random vectors having any shape parameters α_1, α_2 and allowable correlation ρ. In this section we will discuss his general procedure.

If Z, W_1 and W_2 are independent gamma random variables with shape parameters r, δ_1, and δ_2, respectively, and U is an independent uniform [0,1] random variable, and either V = U or V = 1-U; then

$$X_1 = F_n^{-1}(U) + Z + W_1 \qquad \text{(II-C-5)}$$

$$X_2 = F_m^{-1}(V) + Z + W_2$$

where F_n^{-1} and F_m^{-1} are the inverse distribution functions of gamma (n,1) and gamma (m,1), respectively, distribution function. The resulting X_1, X_2 are gamma random variables, with shape parameters

$$\alpha_1 = n + r + \delta_1,$$
$$\alpha_2 = m + r + \delta_2, \tag{II-C-6}$$

respectively. This result follows immediately from the reproducibility property of the gamma distribution and from noting that $F^{-1}(u)$ and $F^{-1}(1-u)$ are each random variables having CDF F. This scheme generalizes the trivariate scheme but brings in the inverse CDF.

The correlation coefficient is defined as

$$\rho_{x_1,x_2} = \frac{E[F_n^{-1}(u)\ F_m^{-1}(v)] - nm + r}{(\alpha_1\alpha_2)^{1/2}} \tag{II-C-7}$$

The remaining problem is to select values of the five parameters n, m, r, δ_1 and δ_2 to obtain the desired marginal distributions and correlation. The following conditions must be satisfied.

$$n + r + \delta_1 = \alpha_1$$

$$m + r + \delta_2 = \alpha_2$$

$$\frac{E[F_n^{-1}(u)\ F_m^{-1}(v)] - (nm) + r}{(\alpha_1 \alpha_2)^{1/2}} = \rho \qquad \text{(II-C-8)}$$

$$H_1,\ H_2,\ r,\ \delta_1,\ \delta_2 \geq 0$$

Since we are using five variables to satisfy three equality conditions, finding a set of parameter values corresponds to finding a feasible solution, rather than an optimal solution, to a nonlinear programming problem.

An efficient solution procedure for determining parameter values is important, since substantial computation is required to determine whether or not conditions (II-C-8) are satisfied for given parameter values. Most of the computation is involved in calculating

$$\rho = \frac{E[F_n^{-1}(u)\ F_m^{-1}(v)] - nm - r}{(\alpha_1 \alpha_2)^{1/2}},$$

since the expected value must be calculated numerically using any one of the following three integrals:

$$\int_0^1 F_n^{-1}(u)\ F_m^{-1}(u)\ du \qquad \text{(II-C-9a)}$$

$$\int_0^\infty F_m^{-1}(F_n(x))\ x^n \exp(-x)\ dx/\Gamma(n) \qquad \text{(II-C-9b)}$$

$$\int_0^\infty F_m^{-1}(F_n(-\ln y))(-\ln y)^n \, dy/\Gamma(n) \qquad \text{(II-C-9c)}$$

if $\rho > 0$. If $\rho < 0$, then replace $F_n^{-1}(u)$ with $F_n^{-1}(1-u)$ in equation (II-C-9a), replace $F_n(x)$ with $1 - F_n(x)$ in equation (II-C-9b), and replace $F_n(-\ln y)$ with $1 - F_n(-\ln y)$ in equation (II-C-9c).

Schmeiser seemed to have best results in terms of a subjective trade off between speed and accuracy, using a 24 point Gaussian method for the integration. He selected the parameter values from the feasible values satisfying conditions (II-C-8) by making the curves of regression $E[X_1|X_2]$ and $E[X_2|X_1]$ behave as desired.

<u>Generating Procedure</u>

1. Generate $X_1 \sim$ gamma $(n,1)$

2. $U \leftarrow F_n(X_1)$

   If $\rho < 0$, $U \leftarrow 1 - u$

3. Generate $Z \sim$ gamma $(r,1)$

4. Generate $W_1 \sim$ gamma $(\delta_1,1)$

5. Generate $W_2 \sim$ gamma $(\delta_2,1)$

6. $X_1 \leftarrow X_1 + Z + W_1$

   $X_2 \leftarrow F_{H_2}^{-1}(U) + Z + W_2$

Based on these procedures he developed a family of algorithms which can provide variates having any theoretically possible correlation ρ. Anyway, for gamma marginal distributions, not all correlations are consistent with particular shape parameter values. Schmeiser shows the obtainable correlations as a function of α_2, given $\alpha_1 = 1$ and 5 as shown in Section II-A. Note that only when $\alpha_1 = \alpha_2$ is it possible to obtain $\rho = 1$. Likewise $\rho = -1$ is not possible except in the limit as α_1 and α_2 tend to infinity. The maximum and minimum possible correlations, given in Moran (1969), occur when

$$X_2 = F_{\alpha_2}^{-1}(F_{\alpha_1}(X_1))$$

and

$$X_2 = F_{\alpha_2}^{-1}(1 - F_{\alpha_1}(X_1)),$$

respectively, where $F_\alpha(X)$ and $F_\alpha^{-1}(U)$ are the cumulative distribution function and inverse CDF, respectively, of the gamma distribution with shape parameter α.

III. GENERAL MIXTURE-TRUNCATION METHOD

Denote by (Y,Z) the bivariate random pair, where each has identical marginal continuous distribution F(x), and denote a general random variable from this distribution by X. The argument is not specific to continuous random variables; this aspect comes in only in the computation of the correlations and can be developed in a parallel fashion for discrete marginal distributions. Let

$$P = \begin{bmatrix} \alpha_1 & 1-\alpha_1 \\ 1-\alpha_2 & \alpha_2 \end{bmatrix},$$

with stationary vector

$$\bar{\pi} = \bar{\pi} P = \left(\frac{1-\alpha_2}{1-\alpha_1+1-\alpha_2}, \frac{1-\alpha_1}{1-\alpha_1+1-\alpha_2}\right) = (\pi_1, \pi_2),$$

and let X_1 be an X truncated to the left of a fixed point x_o, X_2 be an X truncated to the right of x_o, so that

$$F_{X_1}(x) = P[X_1 \le x] = \begin{cases} \dfrac{F(x)}{F(x_o)} & \text{if } x \le x_o \\ 1 & \text{if } x > x_o \end{cases}$$

$$F_{X_2}(x) = P[X_2 \le x] = \begin{cases} 0 & \text{if} \quad x \le x_o \\ \dfrac{F(x) - F(x_o)}{1 - F(x_o)} & \text{if} \quad x > x_o \end{cases}$$

In addition we set $\pi_1 = F(x_o)$, $\pi_2 = 1 - \pi_1$ and choose Y and Z as follows.

i) Choose Y from X_1 with probability π_1 and then choose Z from X_1 with probability α_1, or from X_2 with probability $1 - \alpha_1$.

ii) Choose Y from X_2 with probability π_2, where $\pi_1 + \pi_2 = 1$, and then choose Z from X_1 with probability $1 - \alpha_2$, or from X_2 with probability α_2.

If we choose (Y,Z) as in the above procedure, then we can make the following two theorems.

A. MARGINAL DISTRIBUTIONS

Theorem 1.

The marginal distribution of (Y,Z) becomes F(x) for both Y and Z.

Proof

1. Marginal distribution of Y

By definition Y is the mixture of X_1 and X_2 with probability π_1, π_2 respectively. That is

$$F_Y(x) = \pi_1 F_{X_1}(x) + \pi_2 F_{X_2}(x)$$

$$F_Y(x) = \begin{cases} \pi_1 \dfrac{F(x)}{F(x_o)} + \pi_2 \cdot 0 & \text{if } x \le x_o \\ \pi_1 \cdot 1 + \pi_2 \dfrac{F(x) - F(x_o)}{1 - F(x_o)} & \text{if } x > x_o \end{cases}$$

But since we define $\pi_1 = F(x_o)$; $\pi_2 = 1 - \pi_1 = 1 - F(x_o)$, we have

$$F_Y(x) = \begin{cases} \pi_1 \dfrac{F(x_o)}{\pi_1} = F(x) & \text{if } x \le x_o \\ \pi_1 + \pi_2 \dfrac{F(x) - \pi_1}{\pi_2} = F(x) & \text{if } x > x_o \end{cases}$$

$$= F(x) \quad \text{in all cases.}$$

So Y has the marginal distribution F(x).

2. Marginal distribution of Z

If Y is from X_1, then

$$F_{Z_1}(x) = \alpha_1 F_{X_1}(x) + (1 - \alpha_1) F_{X_2}(x)$$

$$= \begin{cases} \alpha_1 \dfrac{F(x)}{F(x_o)} + (1 - \alpha_1) \cdot 0 & \text{if } x \le x_o \\ \alpha_1 \cdot 1 + (1 - \alpha_1) \dfrac{F(x) - F(x_o)}{1 - F(x_o)} & \text{if } x > x_o \end{cases}$$

If Y is from X_2, then

$$F_{Z_2}(x) = (1-\alpha_2)\, F_{X_1}(x) + \alpha_2\, F_{X_2}(x)$$

$$= \begin{cases} (1-\alpha_2)\,\dfrac{F(x)}{F(x_o)} + \alpha_2 \cdot 0 & \text{if } x \le x_o \\[2ex] (1-\alpha_2)\cdot 1 + \alpha_2\,\dfrac{F(x) - F(x_o)}{1 - F(x_o)} & \text{if } x > x_o \end{cases}$$

So,

$$F_Z(x) = \pi_1\, F_{Z_1}(x) + \pi_2\, F_{Z_2}(x)$$

$$= \begin{cases} \pi_1\alpha_1\,\dfrac{F(x)}{F(x_o)} + \pi_2(1-\alpha_2)\,\dfrac{F(x)}{F(x_o)} & \text{if } x \le x_o \\[2ex] \pi_1(\alpha_1 + (1-\alpha_1)\,\dfrac{F(x) - F(x_o)}{1 - F(x_o)} & \text{if } x > x_o \\[2ex] \quad + \pi_2((1-\alpha_2) + \alpha_2\,\dfrac{F(x) - F(x_o)}{1 - F(x_o)} & \end{cases}$$

and we defined π_1 and π_2 as follows.

$$\pi_1 = \frac{1-\alpha_2}{1-\alpha_1+1-\alpha_2}$$

$$\pi_2 = \frac{1-\alpha_1}{1-\alpha_1+1-\alpha_2}$$

From this, we know

$$1 - \alpha_2 = \frac{\pi_1}{\pi_2} (1 - \alpha_1)$$

If we use this relationship, then $F_Z(x) = F(x)$ in both cases. So Z also has the marginal distribution F(x). The result is a consequence of the fact that $\underline{\pi}$ is defined to be the stationary vector associated with $\underline{P}$.

B. THE PRODUCT-MOMENT CORRELATION

Theorem 2.

The correlation coefficient between Y and Z becomes

$$\rho = \beta M ,$$

where

$$-1 \leq \beta = \alpha_1 - (1 - \alpha_2) \leq 1 ,$$

and

$$M = (\mu_1 - \mu_2)^2 \pi_1 \pi_2 / \sigma_x^2 ,$$

where

$$\mu_1 = \int_0^{x_o} x \, d \frac{F(x)}{F(x_o)}$$

$$\mu_2 = \int_{x_o}^{\infty} x \, d\frac{F(x) - F(x_o)}{1 - F(x_o)}$$

$$\sigma_1^2 = \int_0^{x_o} x^2 \, d\,\frac{F(x)}{F(x_o)} - \mu_1^2$$

$$\sigma_2^2 = \int_{x_o}^{\infty} x^2 \, d\,\frac{F(x) - F(x_o)}{1 - F(x_o)} - \mu_2^2$$

$$\sigma_X^2 = \int_0^{\infty} x^2 \, dF(x) - E[X]$$

$$= \sigma_1^2 \pi_1 + \sigma_2^2 \pi_2 + (\mu_1 - \mu_2)^2 \pi_1 \pi_2$$

Proof

$$\rho_{YZ} = \frac{\text{cov}[Y,Z]}{\sigma_Y \sigma_Z}$$

$$= \frac{E[YZ] - E[Y]\,E[Z]}{\sigma_Y \sigma_Z}$$

Now

$$
\begin{aligned}
E[YZ] &= E_s[E[YZ|Y,Z \varepsilon S]] \\
&= E[YZ|Y \varepsilon X_1, Z \varepsilon X_1]P[Y \varepsilon X_1, Z \varepsilon X_1] \\
&\quad + E[YZ|Y \varepsilon X_1, Z \varepsilon X_2]P[Y \varepsilon X_1, Z \varepsilon X_2] \\
&\quad + E[YZ|Y \varepsilon X_2, Z \varepsilon X_1]P[Y \varepsilon X_2, Z \varepsilon X_1] \\
&\quad + E[YZ|Y \varepsilon X_2, Z \varepsilon X_1]P[Y \varepsilon X_2, Z \varepsilon X_2] \\
&= E[X_1]\ E[X_1]\ \pi_1\alpha_1 + E[X_1]\ E[X_2]\ \pi_1(1-\alpha_1) \\
&\quad + E[X_2]\ E[X_1]\ \pi_2(1-\alpha_2) + E[X_2]\ E[X_2]\ \pi_2\alpha_2 \\
&= \mu_1^2\ \alpha_1\ \pi_1 + \mu_1\ \mu_2(1-\alpha_1)\ \pi_1 \\
&\qquad + \mu_1\ \mu_2(1-\alpha_2)\pi_2 + \mu_2^2\ \alpha_2\pi_2
\end{aligned}
$$

where

$$
\mu_1 = E[X_1], \qquad \mu_2 = E[X_2]
$$

Further,

$$
\begin{aligned}
E[Y] &= E_s[E[Y|Y \varepsilon S]] \\
&= E[Y|Y \varepsilon X_1]P[Y \varepsilon X_1] + E[Y|Y \varepsilon X_2]P[Y \varepsilon X_2] \\
&= E[X_1]\ \pi_1 + E[X_2]\ \pi_2 \\
&= \mu_1\ \pi_1 + \mu_2\ \pi_2 = E[Z]
\end{aligned}
$$

by Theorem 1.

Also by Theorem 1,

$$\sigma_Y^2 = E[Y^2] - (E[Y])^2$$

$$= \sigma_X^2 = \sigma_Z^2$$

If we put together these formulae into

$$\rho_{Y,Z} = \frac{E[YZ] - E[Y]E[Z]}{\sigma_Y \sigma_Z}$$

we get

$$\rho_{Y,Z} = \frac{(\mu_1 - \mu_2)^2 \pi_1 \pi_2 (\alpha_1 - (1 - \alpha_2))}{\sigma_X^2}$$

and let

$$\beta = \alpha_1 - (1 - \alpha_2)$$

$$M = (\mu_1 - \mu_2)^2 \pi_1 \pi_2 / \sigma_X^2$$

then

$$\rho = \beta M .$$

C. GENERAL ALGORITHM

We give here three algorithms for implementing the bivariate mixture-truncation method, which we call the FXO

method, the UXO method, and the TXO method. All of these methods are exactly the same except in how the algorithm chooses x_o, the truncation point, from the x_o range $[x_\ell, x_u]$. The first procedure, called the FXO, choose x_o as a fixed point of x_ℓ and x_u and uses the same x_o during the entire routine. The second procedure, called the UXO, chooses x_o uniformly from $[x_\ell, x_u]$ and repeats this step in every call to the algorithm. The third procedure, called the TXO, is the same as the UXO procedure except in that it uses a triangular distribution instead of uniform. It is necessary to fix these choices of x_o becuase in general there is more than one x_o which will give a bivariate pair (Y,Z) with the given marginal distribution and given correlation. The first procedure, FXO, is defective in terms of their discontinuity of distribution while the second and the third, UXO and TXO, are satisfactory in this respect. The choice of the midpoint of the interval $[x_\ell, x_u]$ for x_o in FXO is based on experience presented later for various marginal distributions. Note that the algorithm described here is inefficient in that it generates the truncated variables X_1 and X_2 by comparing random variables X to x_o until one which is respectively greater than or less than x_o is found. More efficient methods can be found in special cases such as the exponential, but the present algorithm requires only a generation of univariate random variables X without regard to the method used to do this. Of course initialization is required and this is specific to each marginal distribution.

General Mixture-Truncation Method

1. (Initialization)

   i) For given marginal distribution $F(x)$ and correlation coefficient ρ find x_o ranges $[x_\ell, x_u]$

2. Define truncation point x_o

   * FXO method

   i) $X_o = \frac{1}{2}(x_\ell + x_u)$

   * UXO method

   i) Generate a uniform [0,1] random variable V_1

   ii) $x_o = x_\ell + (x_u - x_\ell) * V_1$

   * TXO method

   i) Generate two uniform [0,1] random variables V_1, V_2.

   ii) $X_o = x_\ell + x_1 + x_2$

   where

$$x_m = \frac{1}{2}(x_\ell + x_u)$$

$$x_1 = (x_m - x_\ell) * V_1$$

$$x_2 = (x_u - x_m) * V_2$$

3. Compute parameters value, π_1, π_2, α_1, α_2

4. Choose type for Y

   i) Generate a uniform [0,1] random variable U

   ii) If $U \leq \pi_1$, go to 9

5. Y is an X_2

i) Generate a random variable X from F(x)

ii) If $X > x_o$, set $Y \leftarrow X$ and go to 6

iii) Otherwise return to 5.i)

6. Choose type for Z

i) Set $U \leftarrow ((U - \pi_1)/(1 - \pi_1))$

ii) If $U \leq 1 - \alpha_2$, go to 8

7. Z is an X_2

i) Generate a random variable X from F(x)

ii) If $X > x_o$, set $Z \leftarrow X$ and go to 11

iii) Otherwise return to 7.i)

8. Z is on X_1

i) Generate a random variable X from F(x)

ii) If $X \leq x_o$, set $Z \leftarrow X$ and go to 11

iii) Otherwise return to 8.i)

9. Y is an X_1

i) Generate a random variable X from F(x)

ii) If $X \leq x_o$, set $Y \leftarrow X$ and go to 10

iii) Otherwise return to 9.i)

10. Choose type for Z

i) Set $U \leftarrow U/\pi_1$

ii) If $U \leq \alpha_1$, go to 8

iii) Otherwise go to 7

11. Deliver (Y,Z) and go to 4 for the FXO method, or go to 2 for the UXO and TXO method until a sufficient number of random vectors are obtained.

D. BIVARIATE DISTRIBUTION FUNCTIONS

From Theorem 1, in Section III-A, we know that if Y is from X_1, then

$$F_Z(z|y) = \alpha_1 F_{X_1}(z) + (1-\alpha_1) F_{X_2}(z)$$

$$= \begin{cases} \alpha_1 \dfrac{F(z)}{F(x_o)} & \text{if } z \leq x_o \\[2ex] \alpha_1 + (1-\alpha_1) \dfrac{F(z) - F(x_o)}{1 - F(x_o)} & \text{if } z > x_o \end{cases}$$

and if Y is from X_2, then

$$F_Z(z|y) = (1-\alpha_2) F_{X_1}(z) + \alpha_2 F_{X_2}(z)$$

$$= \begin{cases} (1-\alpha_2) \dfrac{F(z)}{F(x_o)} & \text{if } z \leq x_o \\[2ex] (1-\alpha_2) + \alpha_2 \dfrac{F(z) - F(x_o)}{1 - F(x_o)} & \text{if } z > x_o \end{cases}$$

By using these we can define the bivariate distribution function as follows.

$$F(y,z) = P[Y \le y, Z \le z]$$

$$= \int_{-\infty}^{y} P[Z \le z | Y = u] \, dP[Y \le u]$$

where

$$P[Y \le u] = F(u)$$

$$P[Z \le z | Y = u] = \begin{cases} \alpha_1 \dfrac{F(z)}{F(x_o)} & \text{if } u \le x_o,\ z \le x_o \\ \alpha_1 + (1-\alpha_1) \dfrac{F(z) - F(x_o)}{1 - F(x_o)} & \text{if } u \le x_o,\ z > x_o \\ (1-\alpha_2) \dfrac{F(z)}{F(x_o)} & \text{if } u > x_o,\ z \le x_o \\ (1-\alpha_2) + \alpha_2 \dfrac{F(z) - F(x_o)}{1 - F(x_o)} & \text{if } u > x_o,\ z > x_o \end{cases}$$

So, if we put these together, integrating with respect to $dP[Y \le u]$, we get the final result:

$$P[Y \le y, Z \le z] = \begin{cases} \dfrac{\alpha_1}{\pi_1} F(z)F(y) & \text{if } y \le x_o,\ z \le x_o \quad \text{(III-D-1)} \\[2ex] \{\alpha_1 + \dfrac{(1-\alpha_1)}{\pi_2}[F(z)-F(x_o)]\}F(y) & \text{if } y \le x_o,\ z > x_o \quad \text{(III-D-2)} \\[2ex] \{\alpha_1 + \dfrac{(1-\alpha_1)}{\pi_2}[F(y)-F(x_o)]\}F(z) & \text{if } y > x_o,\ z \le x_o \quad \text{(III-D-3)} \\[2ex] \alpha_1\pi_1 + (1-\alpha_1)[F(z)-F(x_o)]\dfrac{\pi_1}{\pi_2} & \quad \text{(III-D-4)} \end{cases}$$

$$+\{(1-\alpha_2) + \frac{\alpha_2}{\pi_2}[F(z)-F(x_o)]\}[F(y)-F(x_o)] \quad \text{if } y > x_o,\ z > x_o$$

For example, the expression (III-D-4) is obtained as

$$F(y,z) = \int_{-\infty}^{y} P[Z \le z \mid Y = u]\ dF(u) \qquad y > x_o, \quad z > x_o$$

$$= \int_{-\infty}^{x_o} P[Z \le z \mid Y = u \le x_o]\ dF(u)$$

$$+ \int_{x_o}^{y} P[Z \le z \mid Y = u > x_o]\ dF(u)$$

$$= \{\alpha_1 + \frac{(1-\alpha_1)}{\pi_2}[F(z) - F(x_o)]\}\ F(x_o)$$

$$+ \quad (1-\alpha_2) + \frac{\alpha_2}{\pi_2}[F(z) - F(x_o)]\}[F(y) - F(x_o)]$$

It is easily seen from (III-D-2) that when $z \to \infty$, $P[Y \le y, Z \le \infty] = F(y)$; from (III-D-3) that as $y \to \infty$,

$P[Y \le \infty, Z \le z] = F(z)$ and from (III-D-4) that as $y \to \infty$,

$P[Y \le \infty, Z \le z] = F(z)$ and that $z \to \infty$, $P[Y \le y, Z \le \infty] = F(y)$.

In particular from (III-D-4) we have that, as $y \to \infty$,

$$F(y,z) \to \alpha_1 F(x_o) + (1-\alpha_2)[F(z) - F(x_o)] + (1-\alpha_2)\pi_2$$

$$+ \alpha_2 [F(z) - F(x_o)]$$

$$= \alpha_1 F(x_o) + (1-\alpha_2)\pi_2 + [F(z) - F(x_o)]$$

$$= F(z) + (1-\alpha_2)\pi_2 - (1-\alpha_1)\pi_1 = F(z)$$

where at the last step we used the facts that $F(x_o) = \pi_1$ and $\pi_1(1-\alpha_1) = \pi_2(1-\alpha_2)$. If $F(x)$ is absolutely continuous with probability density function $f(x)$ then the joint p.d.f. for the bivariate pair (Y,Z) is

$$f(y,z) = \begin{cases} \dfrac{\alpha_1(1-\alpha_1)}{\pi_2(1-\alpha_2)} f(y)\, f(z) & \text{if } y \le x_o,\ z \le x_o \quad \text{(III-D-5)} \\[2ex] \dfrac{1-\alpha_1}{\pi_2} f(y)\, f(z) & \text{if } y \le x_o,\ z > x_o \quad \text{(III-D-6)} \\[2ex] \dfrac{1-\alpha_1}{\pi_2} f(y)\, f(z) & \text{if } y > x_o,\ z \le x_o \quad \text{(III-D-7)} \\[2ex] \dfrac{\alpha_2}{\pi_2} f(y)\, f(z) & \text{if } y > x_o,\ z > x_o \quad \text{(III-D-8)} \end{cases}$$

Note that there is a discontinuity in the density function as one crosses the boundaries of the four quadrants defined by the lines $y = x_o$; $z = x_o$. The density is the same in the first and third quadrants. The multipliers of $f(y)f(z)/\pi_2$ are the same in all four quadrants iff there is independence. This occurs when $(1 - \alpha_1) = \alpha_2$ so that $\pi_1 = \alpha_2$ and $f(y,z) = f(y)f(z)$ for the whole range of y and z.

IV. BIVARIATE UNIFORM GENERATOR

The uniform random variable is a continuous random variable with probability density function which is constant over the interval (a,b) and zero otherwise; the density function

$$f(x) = \begin{cases} \frac{1}{b-a}; & \text{if } a \leq x \leq b, \\ 0; & \text{otherwise}; \end{cases}$$

$$\text{mean} = E[X] = \frac{b+a}{2};$$

$$\text{variance} = VAR[X] = \frac{(b-a)^2}{12}.$$

We note that if U has a uniform distribution over [0,1] then $X = a + (b-a)U$ has a uniform distribution over [a,b]. So we will only be concerned with algorithms for generating uniform [0,1] distributions.

Two algorithms for generating uniform bivariate pairs are given in Section II, Gaver's and Lawrance and Lewis's, and since they have relatively smooth distributed functions and are simple to generate, they are probably preferable to the uniform bivariate random variable generated by the mixture-truncation method unless x_o is taken to be random (of course the correlation for the Gaver bivariate uniform is difficult

to compute). However the development in this section of the uniform mixture-truncation method does help to fix ideas on the problem of initialization and determination of the possible range of correlations attainable in other cases.

To use the mixture-truncation method for generating bivariate random vector (Y,Z) whose marginal distribution is uniform over [0,1] identically and has given correlation ρ, we should use uniform [0,1] distribution functions as F(x). Then X_1, which is X constrained to be on the left side of the truncation point x_o, becomes uniform over $[0,x_o]$ and X_2, which is constrained to be on the right side of x_o becomes uniform over $[x_o,1]$.

A. DETERMINATION OF PARAMETERS IN THE UNIFORM MIXTURE-TRUNCATION METHOD

Because X_1 is uniform $[0,x_o]$ and x_2 is uniform $[x_o,1]$,

$$E[X_1] = \mu_1 = \frac{x_o}{2}$$

$$VAR[X_1] = \sigma_1^2 = \frac{x_o^2}{12}$$

$$E[X_2] = \mu_2 = \frac{1+x_o}{2}$$

$$VAR[X_2] = \sigma_2^2 = \frac{(1-x_o)^2}{12}$$

and

$$\pi_1 = F(x_o) = \frac{1-\alpha_2}{1-\alpha_1+1-\alpha_2} = x_o , \qquad \text{(IV-A-1a)}$$

$$\pi_2 = 1 - F(x_o) = \frac{1 - \alpha_1}{1 - \alpha_1 + 1 - \alpha_2} = 1 - x_o \qquad \text{(IV-A-1b)}$$

If we use these formulas in Theorem 2 of Section III, we get

$$M = 3\, x_o(1 - x_o)$$

$$\beta = \frac{\alpha_1 - x_o}{1 - x_o}$$

$$\rho = 3\, x_o(\alpha_1 - x_o) \qquad \text{(IV-A-2)}$$

Then from (IV-A-2), (IV-A-1a) and (IV-A-1b)

$$\alpha_1 = \frac{\rho}{3x_o} + x_o \qquad \text{(IV-A-3a)}$$

$$\alpha_2 = 1 - \frac{\pi_1}{\pi_2}(1 - \alpha_1)$$

$$= 1 - \frac{x_o^2 + x_o}{1 - x_o} + \frac{\rho}{3(1 - x_o)} \qquad \text{(IV-A-3b)}$$

Furthermore, α_1 and α_2 are probabilities so they have to satisfy the following conditions:

$$0 \leq \alpha_1 \leq 1,$$

$$0 \leq \alpha_2 \leq 1 .$$

By solving these two inequality equations, we can get the feasible range of x_o for given ρ as follows.

$$x_\ell = \text{Lower bound of } x_o$$

$$= \begin{cases} 1/2 - \sqrt{1/4 - (1/3)\rho} & \text{if } \rho > 0 \\ \sqrt{-\rho/3} & \text{if } \rho < 0 \end{cases} \qquad \text{(IV-A-4)}$$

$$x_u = \text{Upper bound of } x_o$$

$$= \begin{cases} 1/2 + \sqrt{1/4 - (1/3)\rho} & \text{if } \rho > 0 \\ 1 - \sqrt{-\rho/3} & \text{if } \rho < 0 \end{cases} \qquad \text{(IV-A-5)}$$

After finding x_ℓ and x_u, choose x_o within the range $[x_\ell, x_u]$ by way of either the FXO, UXO, or TXO method. And by formulas (IV-A-1a), (IV-A-1b), (IV-A-3a) and (IV-A-3b), we can get π_1, π_2, α_1 and α_2. Also we can expect some limitations in ρ because of x_o. From

$$M = 3\, x_o(1 - x_o)$$

$$-1 \leq \beta = \alpha_1 - (1 - \alpha_2) \leq 1$$

$$\rho = \beta M$$

We can compute that

$$\rho_{max} = 3/4 \quad \text{if} \quad \beta = 1, \quad x_\ell \leq x_o = 1/2 \leq x_u$$

$$\rho_{min} = -3/4 \quad \text{if} \quad \beta = -1, \quad x_\ell \leq x_o = 1/2 \leq x_u$$

In another way, note that if one chooses $x_o = 1/2$ and $\alpha_2 = \alpha_1 = 1$, from (IV-A-2) we attain the highest allowable correlation $\rho = 3/4$, and when $\alpha_1 = 0$, $\rho = -3/4$, the lowest attainable correlation.

It turns out that for this choice of $x_o = 1/2$ (in the FXO method), the bivariate uniform distribution is also about the smoothest, as can be seen in a later section (IV-D), and from the bivariate distribution.

B. GENERATING PROCEDURE

We developed here all of the three procedures, the FXO method, the UXO method, the TXO method for generating bivariate random vectors whose marginal distributions are uniform over [0,1] and correlation coefficient is ρ. In this algorithm we used direct generating procedures for X_1 and X_2 instead of comparing random variables X to x_o until one which is respectively greater than or less than x_o is found.

Uniform Mixture-Truncation Method

1. (Initialization)

i) For given $-3/4 < \rho < 3/4$, find x_ℓ and x_u

$$x_\ell = \begin{cases} 1/2 - \sqrt{1/4 - 1/3\,\rho} & \text{if } \rho > 0 \\ \sqrt{-\,\rho/3} & \text{if } \rho < 0 \end{cases}$$

$$x_u = \begin{cases} 1/2 + \sqrt{1/4 - 1/3\,\rho} & \text{if } \rho > 0 \\ 1 - \sqrt{-\rho/3} & \text{if } \rho < 0 \end{cases}$$

2. Define truncation point x_o.

* FXO method

i) $x_o = (x_\ell + x_u)/2$

* UXO method

i) Generate a uniform [0,1] random variable U_1

ii) $x_o = x_\ell + (x_u - x_\ell) * U_1$

* TXO method

i) Generate two uniform [0,1] random variables V_1, V_2

ii) $x_o = x_\ell + x_1 + x_2$

where

$$x_1 = (x_m - x_\ell) * V_1$$

$$x_2 = (x_u - x_m) * V_2$$

$$x_m = (x_u + x_\ell)/2$$

3. Compute π_1, π_2, α_1 and α_2 as

$$\pi_1 = F(x_o) = x_o$$

$$\pi_2 = 1 - \pi_1$$

$$\alpha_1 = \frac{\rho}{3} x_o + x_o$$

$$\alpha_2 = 1 - \frac{\pi_1}{\pi_2}(1 - \alpha_1)$$

4. Choose type for Y

   i) Generate a uniform [0,1] random variable U

   ii) If $U \leq \pi_1$ go to 9

5. Y is on X_2

   i) Generate a uniform [0,1] random variable W_1

   ii) $Y \leftarrow x_o + (1.0 - x_o) * W_1$

6. Choose type for Z

   i) Set $U \leftarrow ((U - \pi_1)/(1 - \pi_1))$

   ii) If $U \leq 1 - \alpha_2$, go to 8

7. Z is an X_2

   i) Generate a uniform [0,1] random variable W_2

   ii) $Z \leftarrow x_o + (1.0 - x_o) * W_2$

   iii) Go to 11

8. Z is an X_1

   i) Generate a uniform [0,1] random variable W_2

   ii) $Z \leftarrow x_o * W_2$

   iii) Go to 11

9. Y is an X_1

   i) Generate a uniform [0,1] random variable W_1

   ii) $Y \leftarrow x_o * W_1$

10. Choose type for Z

   i) Set $U \leftarrow U/\pi_1$

   ii) If $U \leq \alpha_1$ go to 8

   iii) Otherwise go to 7

11. Deliver (Y,Z) and go to 4, for the FXO method, or go to 2 for the UXO method and the TXO method until a sufficient number of random vectors are obtained.

The programs are listed in the appendix and the scatter plot and bivariate histogram of resulting random vectors are shown in Section IV-D.

C. REGRESSION OF Z ON Y FOR GIVEN ρ

Schmeiser (1979) in developing a bivariate gamma method fixed the free parameter by specifying the regression of Z and Y and vice-versa. We do this now for the uniform case with fixed x_o and uniformly distributed x_o. First in fixed x_o case we have, for $y \leq x_o$

$$E[Z|Y=y, y \leq x_o] = \alpha_1 E[X_1] + (1-\alpha_1) E[X_2]$$

$$= 1/2 - \frac{1}{6x_o} \rho$$

and for $y > x_o$

$$E[Z|Y=y, y > x_o] = (1-\alpha_2) E[X_1] + \alpha_2 E[X_2]$$

$$= 1/2 + \rho \frac{1}{6(1-x_o)}$$

These are the step function and clearly not linear in y. If x_o has uniform distribution over $[x_\ell, x_u]$, we have the following results. These results are dependent on Y values.

If $y \leq x_\ell$, then we have

$$E[Z|Y=y] = \int_{x_\ell}^{x_u} E[Z|Y=y, X=x_o, y \leq x_o] f(x_o) dx_o$$

$$= \int_{x_\ell}^{x_u} \left(\frac{1}{2} - \frac{1}{6x_o} \rho\right) \frac{1}{x_u - x_\ell} dx_o$$

$$E[Z|Y=y] = \frac{1}{x_u - x_\ell}\left(\frac{1}{2} x_u - \frac{1}{2} x_\ell - \frac{\rho}{6} \ln \frac{x_u}{x_\ell}\right)$$

If $x_\ell \leq y \leq x_u$, then we have

$$E[Z|Y=y] = \int_{x_\ell}^{y} E[Z|Y=y,\ X=x_o,\ y > x_o] f(x_o)\ dx_o$$

$$+ \int_{y}^{x_u} E[Z|Y=y,\ X=x_o,\ y \leq x_o]\ f(x_o)\ dx_o$$

$$= \int_{x_\ell}^{y} \left(\frac{1}{2} + \frac{\rho}{6(1-x_o)}\right) f(x_o)\ dx_o$$

$$+ \int_{y}^{x_o} \left(\frac{1}{2} - \frac{\rho}{6x_o}\right) f(x_o)\ dx_o$$

$$= \frac{1}{x_u - x_\ell}\left(\frac{1}{2} x_u - \frac{1}{2} x_\ell - \frac{\rho}{6} \ln \frac{1-y}{y} \frac{x_u}{1-x_\ell}\right)$$

If $y > x_\ell$, then we

$$E[Z|Y=y] = \int_{x_\ell}^{x_u} E[Z|Y=y,\ X=x_o,\ y > x_o]\ f(x_o)\ dx_o$$

$$= \int_{x_\ell}^{x_u} \left(\frac{1}{2} + \frac{\rho}{6(1-x_o)}\right) f(x_o)\ dx_o$$

$$= \frac{1}{x_u - x_\ell}\left(\frac{1}{2} x_u - \frac{1}{2} x_\ell - \frac{\rho}{6} \ln \frac{1-x_u}{1-x_\ell}\right)$$

From the results we can see that if we use uniform distribution for x_o we can get smoother regression functions if $x_\ell \leq y \leq x_u$ although we still have step functions for the $y \leq x_\ell$ and $y \geq x_u$ cases.

D. SIMULATION RESULTS

We will show here the scatter plots and the bivariate histograms of the mixture-truncation bivariate uniform (0,1) random vectors with correlation $\rho = 0.3$ and $\rho = -0.3$ by the FXO, UXO and TXO methods in Figures (IV-a), (IV-b) and (IV-c), respectively.

In addition, we will show the scatter plot and bivariate histogram of sample size 2000 from Gaver's transformation and Lawrance-Lewis method in Figures (IV-e) and (IV-d), respectively. The Gaver method does not have known correlation but the value of p to give ρ in the exponential case is used.

As we mentioned earlier, the FXO method has some discontinuity at the truncation point. But the UXO and TXO methods generate relatively smooth continuous distributions. In this respect, we say that the FXO is defective and the UXO and the TXO are relatively satisfactory.

The computed correlation from subroutine BIVHST (Bivariate histogram) is a little different from given correlation. But this can be assumed as a sampling error. To check this error, we simulated 10 times with given correlation $\rho = 0.1$. From this simulation we get the following results:

$\bar{\rho}$ = mean of computed correlation = 0.105

VAR[ρ] = variance of computed correlation = 0.0004

$\sigma[\bar{\rho}]$ = standard deviation of mean = 0.0064

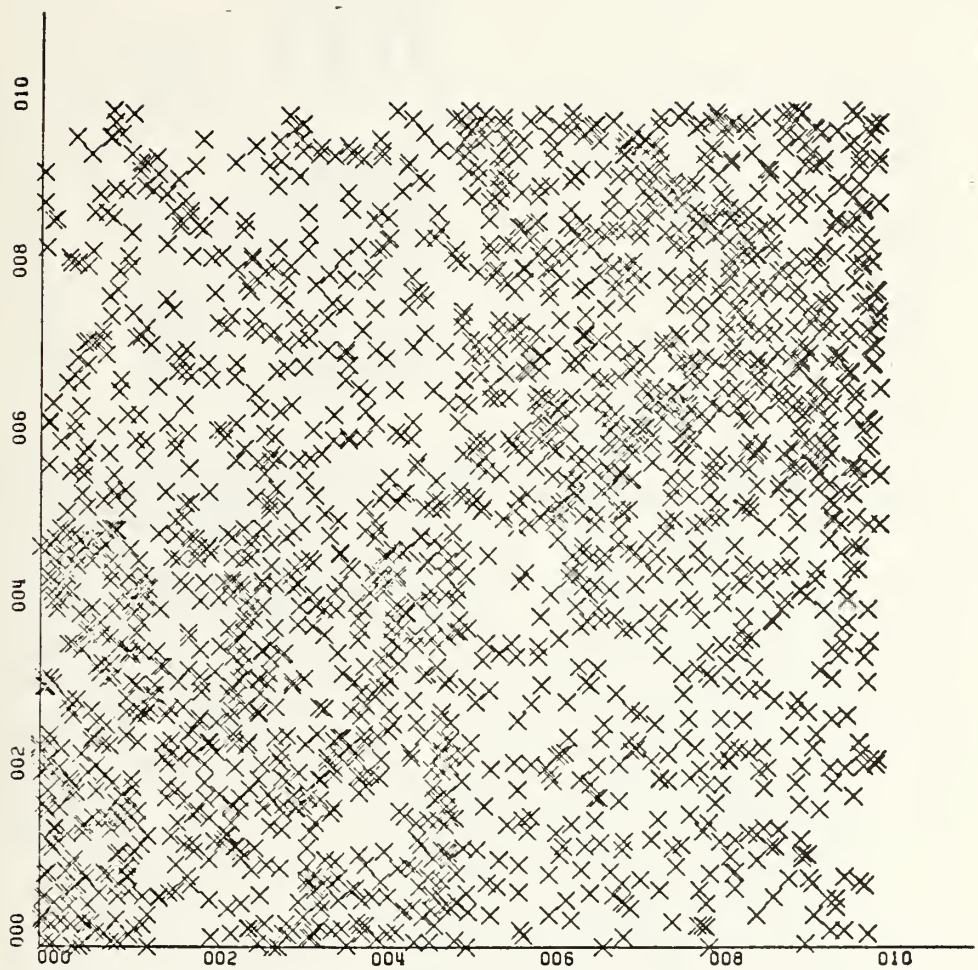


Figure IV-al. Scatter plot for sample of size 2000 from mixture-truncation bivariate uniform with $\rho = 0.3$. Here x_o is fixed at the midpoint between the lower and upper bounds.

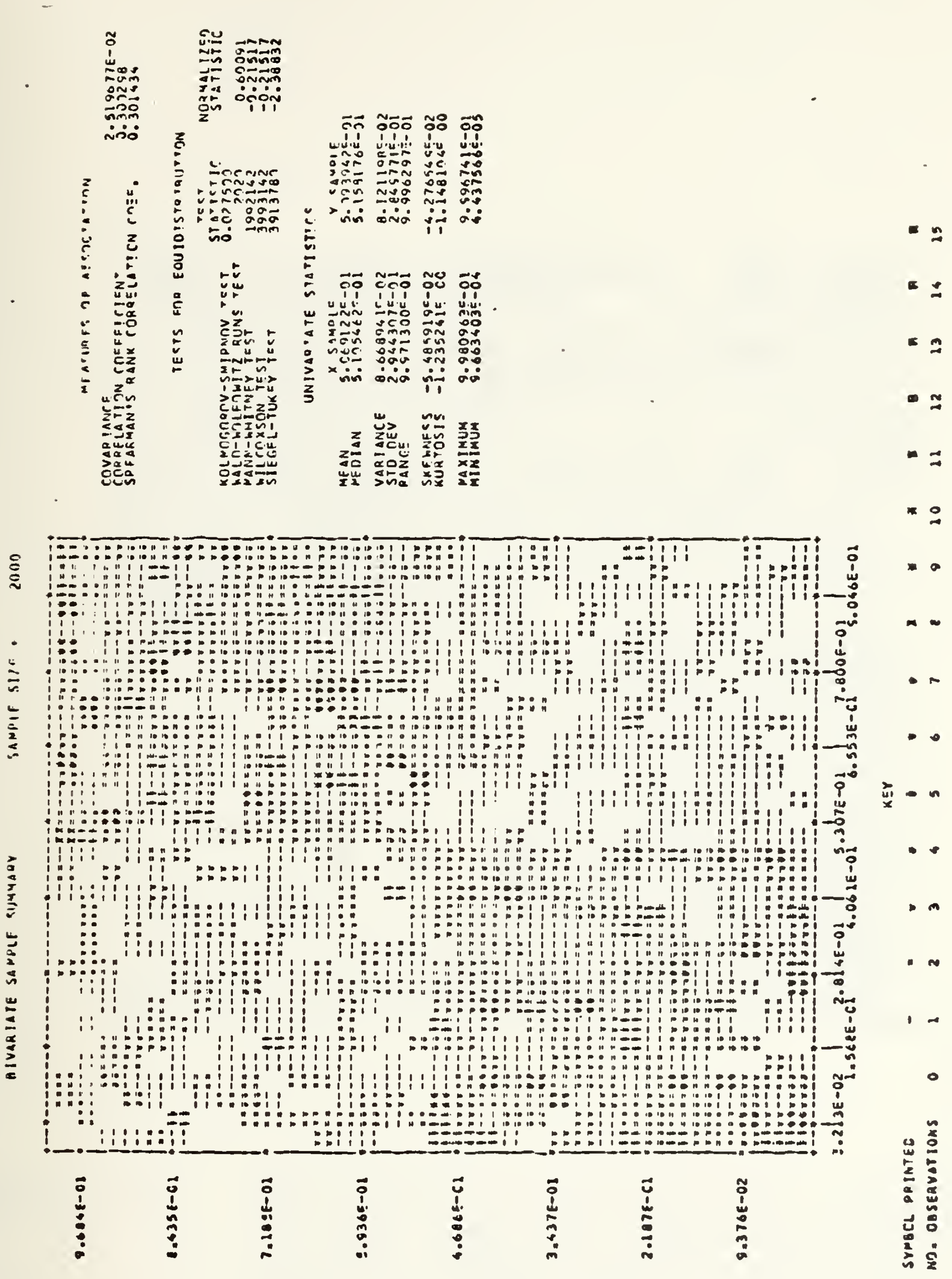


Figure IV-a2. Bivariate historgram for sample of size 2000 from mixture-truncation bivariate uniform with $\rho = 0.3$. Here x_0 is fixed at the midpoint between the lower and the upper bounds.

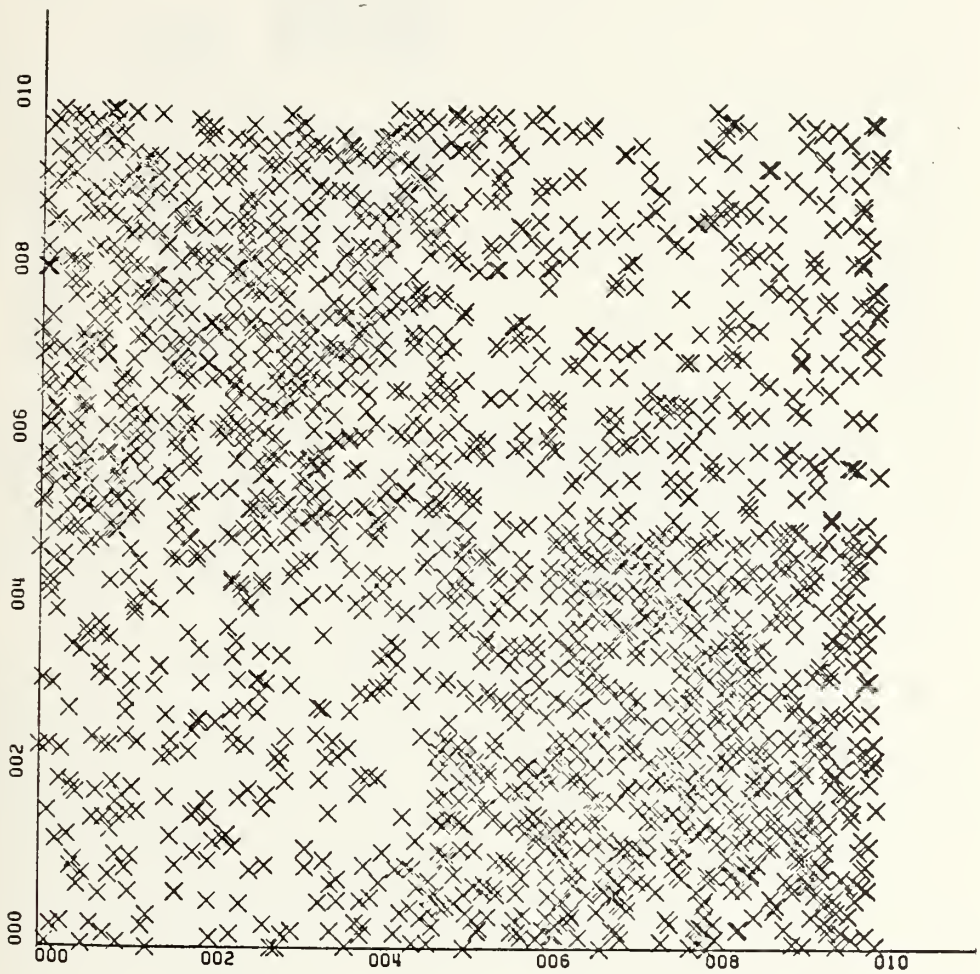


Figure IV-a3. Scatter plot for sample of size 2000 from mixture-truncation bivariate uniform with $\rho = 0.3$. Here x_o is fixed at the midpoint between the lower and the upper bounds.

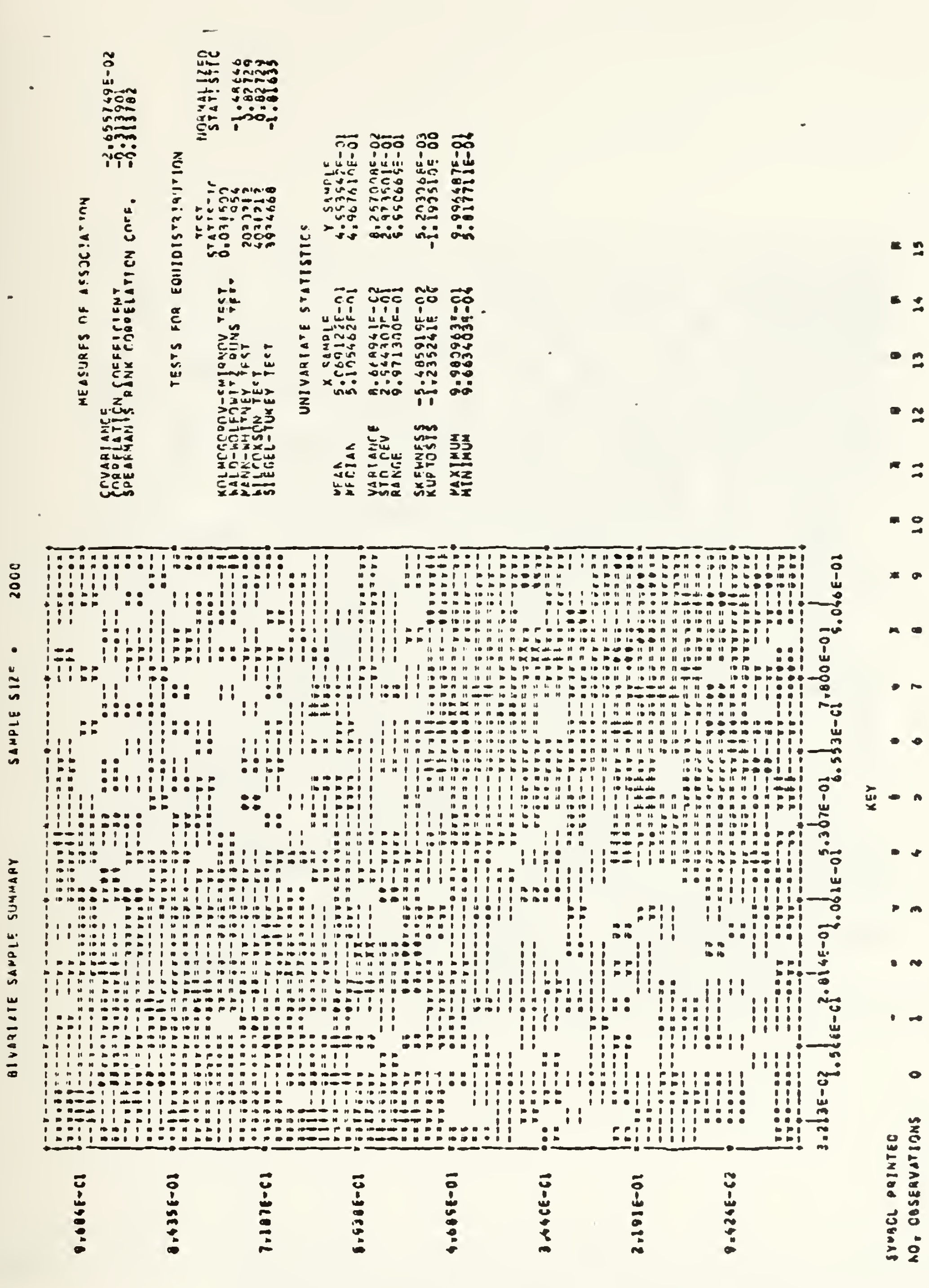


Figure IV-a4. Bivariate histogram for sample of size 2000 from mixture-truncation bivariate uniform with $\rho = -0.3$. Here x_0 is fixed at the midpoint between the lower and the upper bounds.

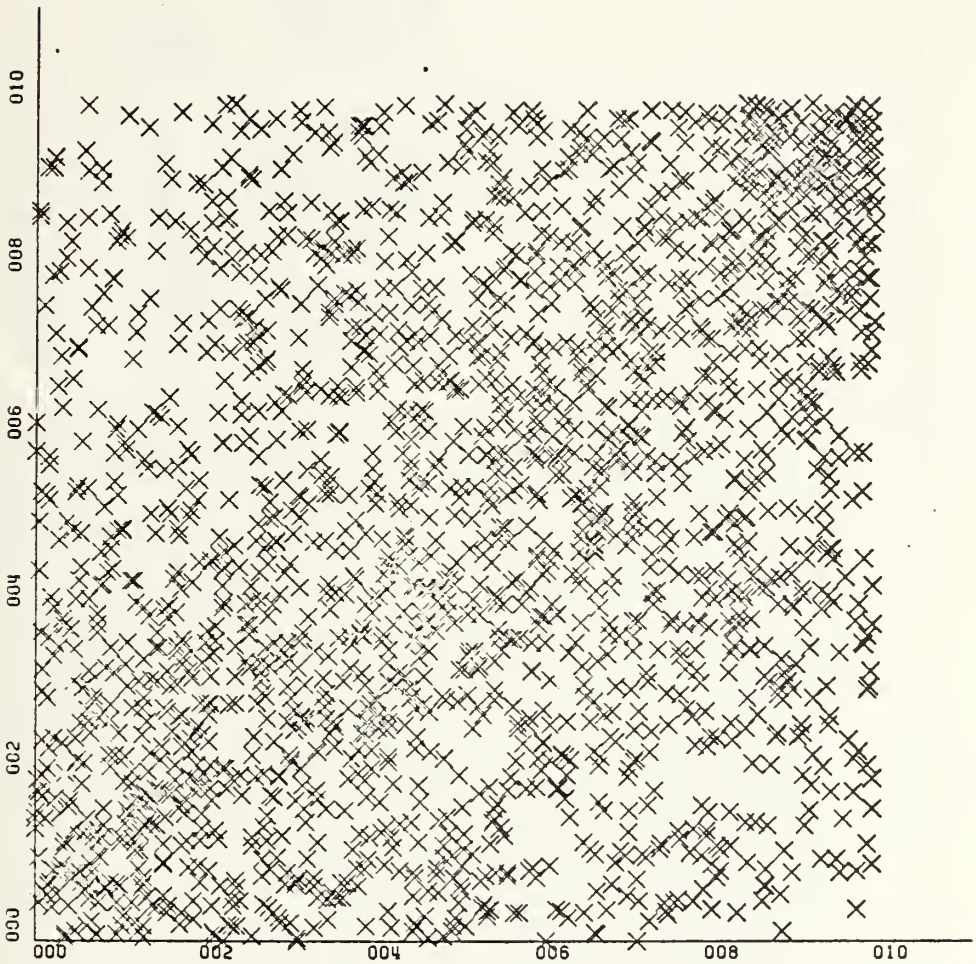


Figure IV-bl. Scatter plot for sample of size 2000 from mixture-truncation bivariate uniform with $\rho = 0.3$. Here x_o is taken to be uniformly distributed between the lower and the upper bounds.

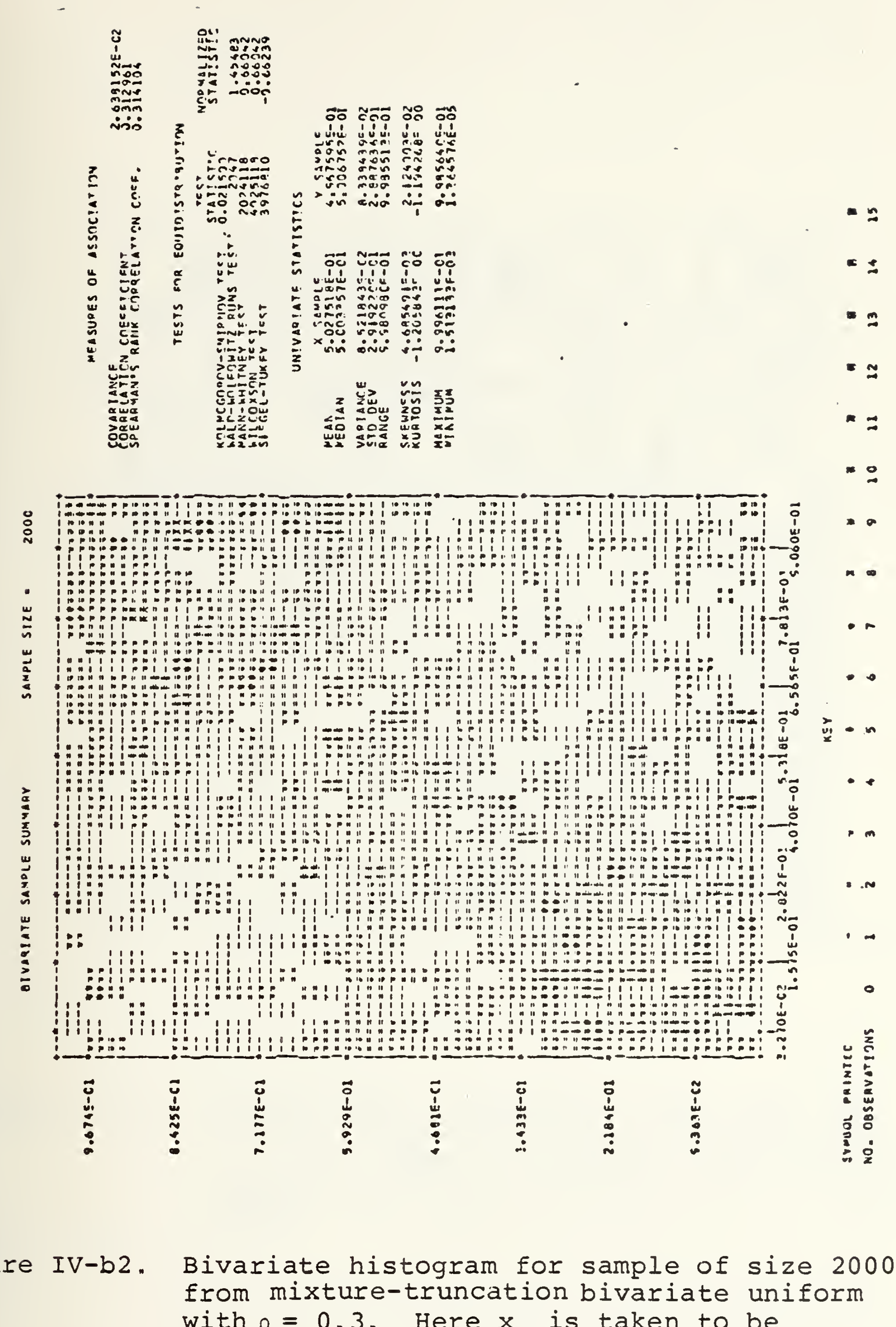


Figure IV-b2. Bivariate histogram for sample of size 2000 from mixture-truncation bivariate uniform with $\rho = 0.3$. Here x_0 is taken to be uniformly distributed between the lower and the upper bounds.

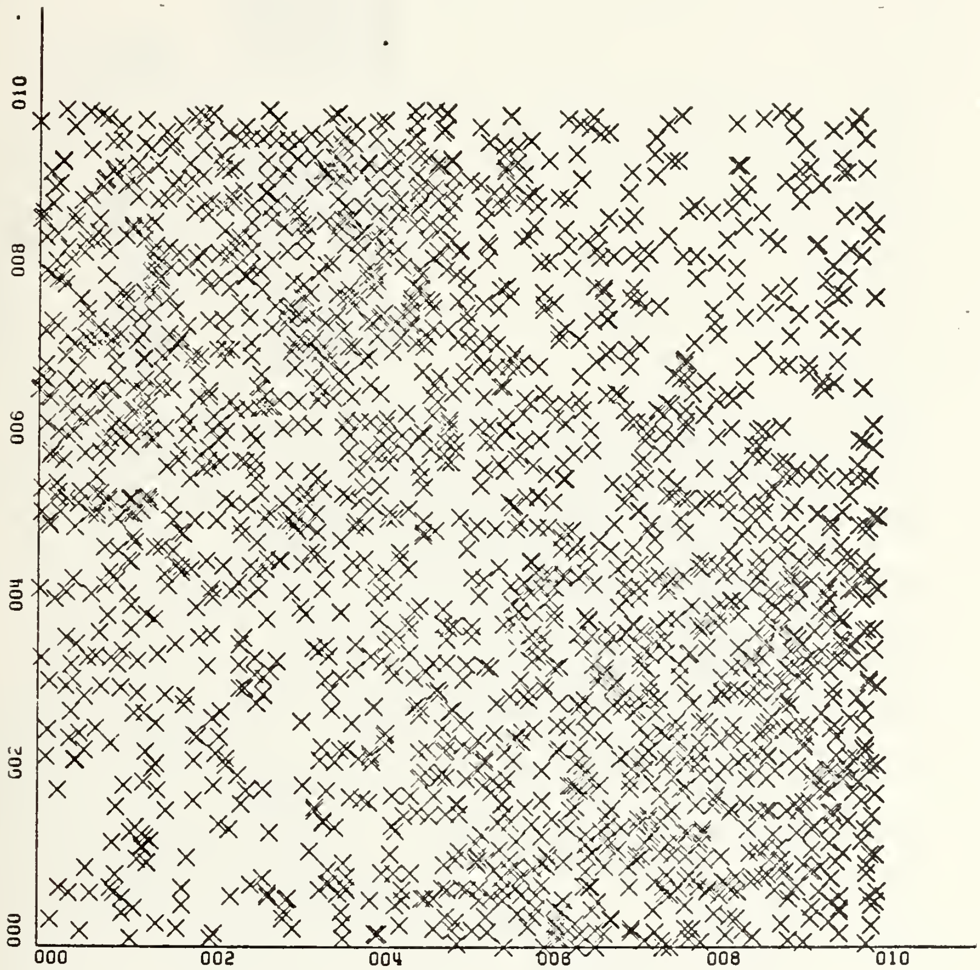


Figure IV-b3. Scatter plot for sample of size 2000 from mixture-truncation bivariate uniform with $\rho = -0.3$. Here x_o is taken to be uniformly distributed between the lower and the upper bounds

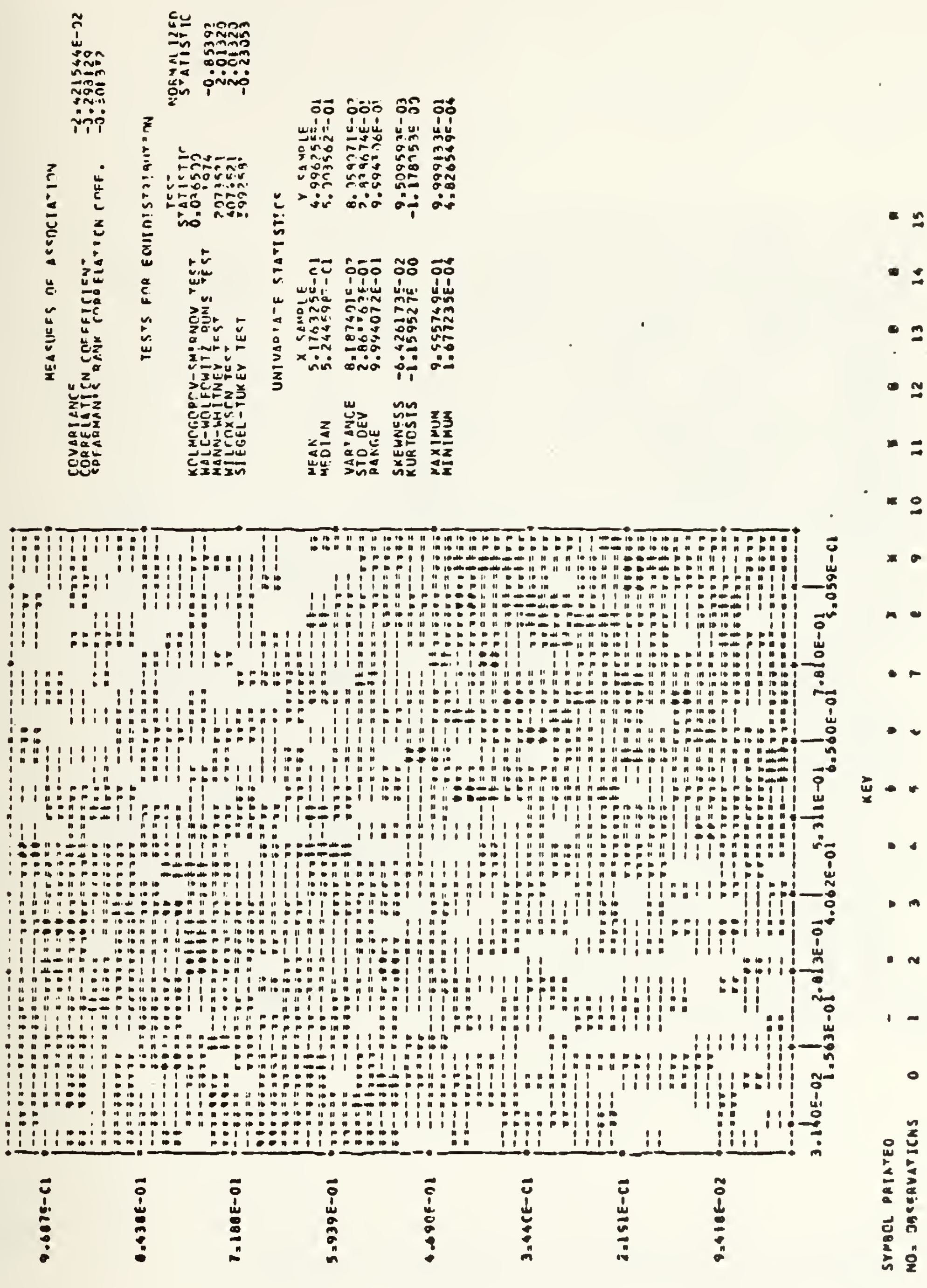


Figure IV-b4. Bivariate histogram for sample of size 2000 from mixture-truncation bivariate uniform with $\rho = -0.3$. Here x_0 is taken to be uniformly distributed between the lower and the upper bounds.

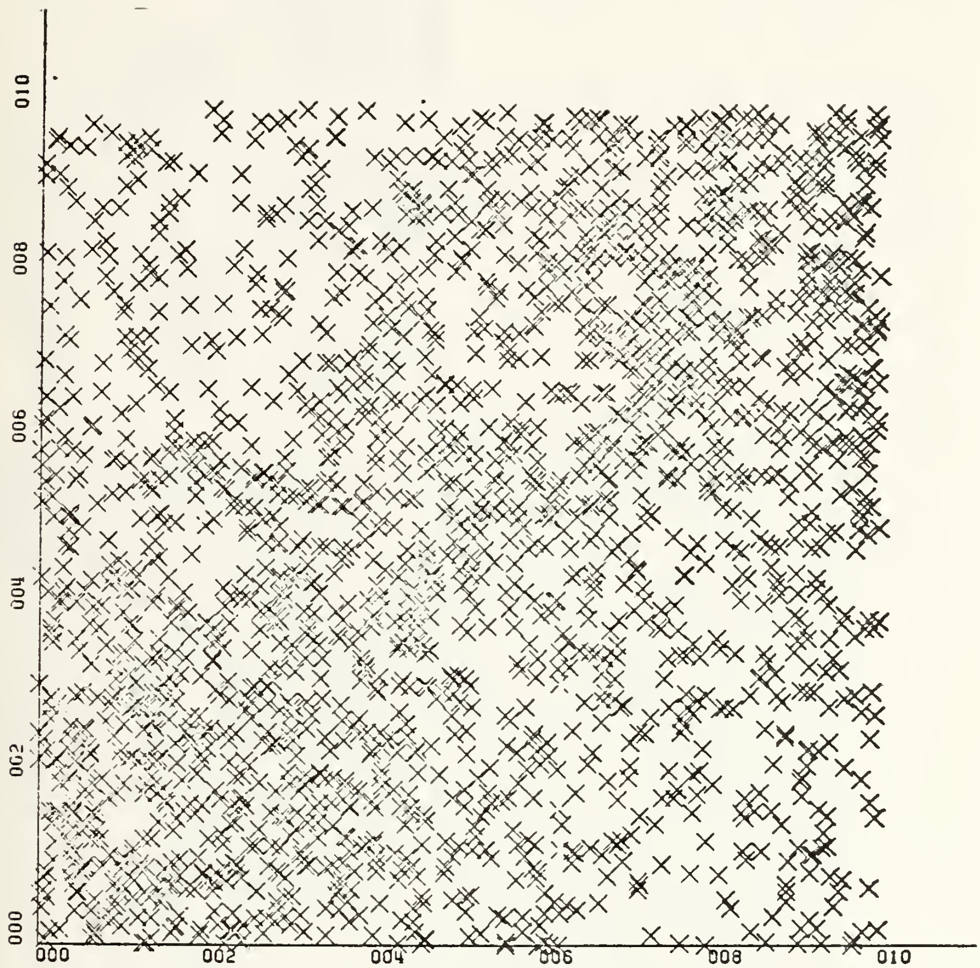


Figure IV-c1. Scatter plot for sample of size 2000 from mixture-truncation bivariate uniform with $\rho = 0.3$. Here x_o has triangular distribution between the lower and the upper bounds.

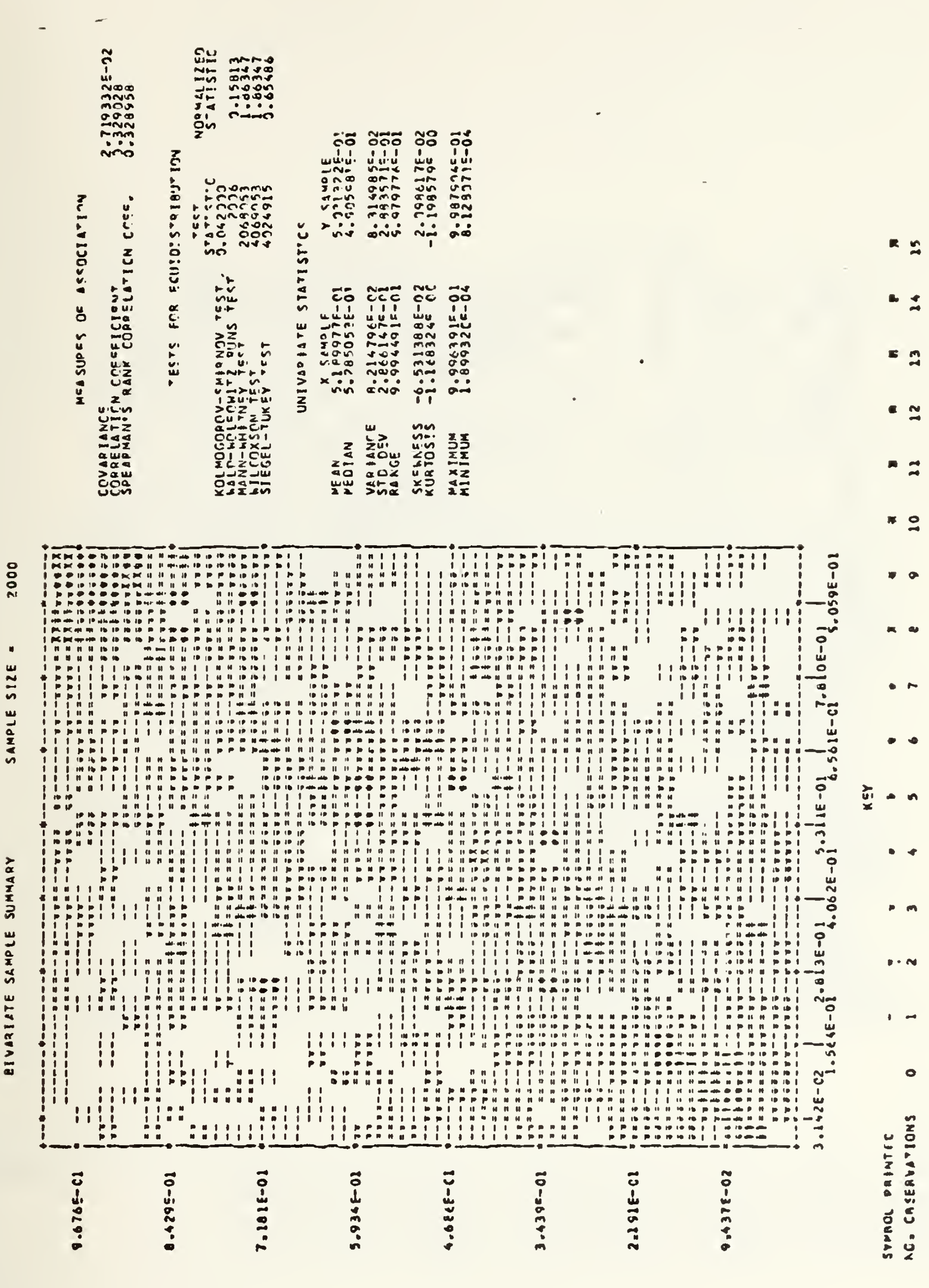


Figure IV-c2. Bivariate histogram for sample of size 2000 from mixture-truncation bivariate uniform with $\rho = 0.3$. Here x_o has triangular distribution between the lower and the upper permissible bounds.

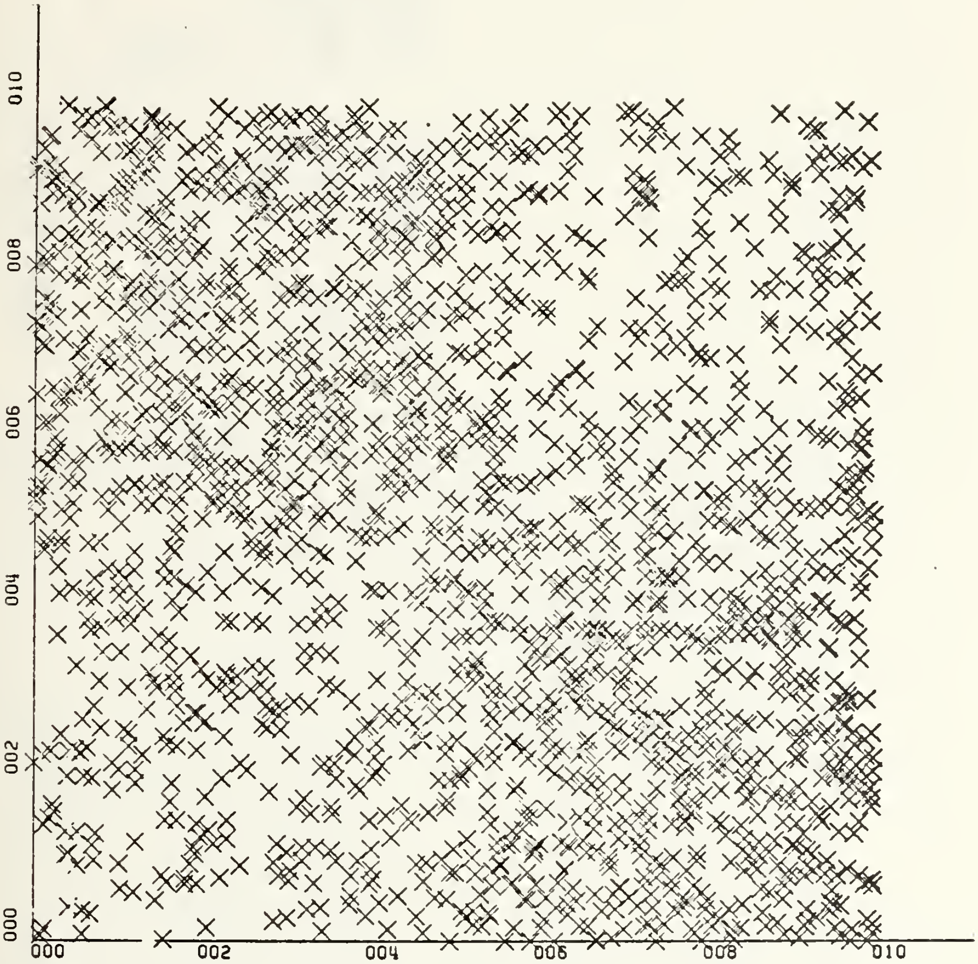


Figure IV-c3. Scatter plot for sample of size 2000 from mixture-truncation bivariate uniform with $\rho = -0.3$. Here x_0 has triangular distribution between the lower and the upper bounds.

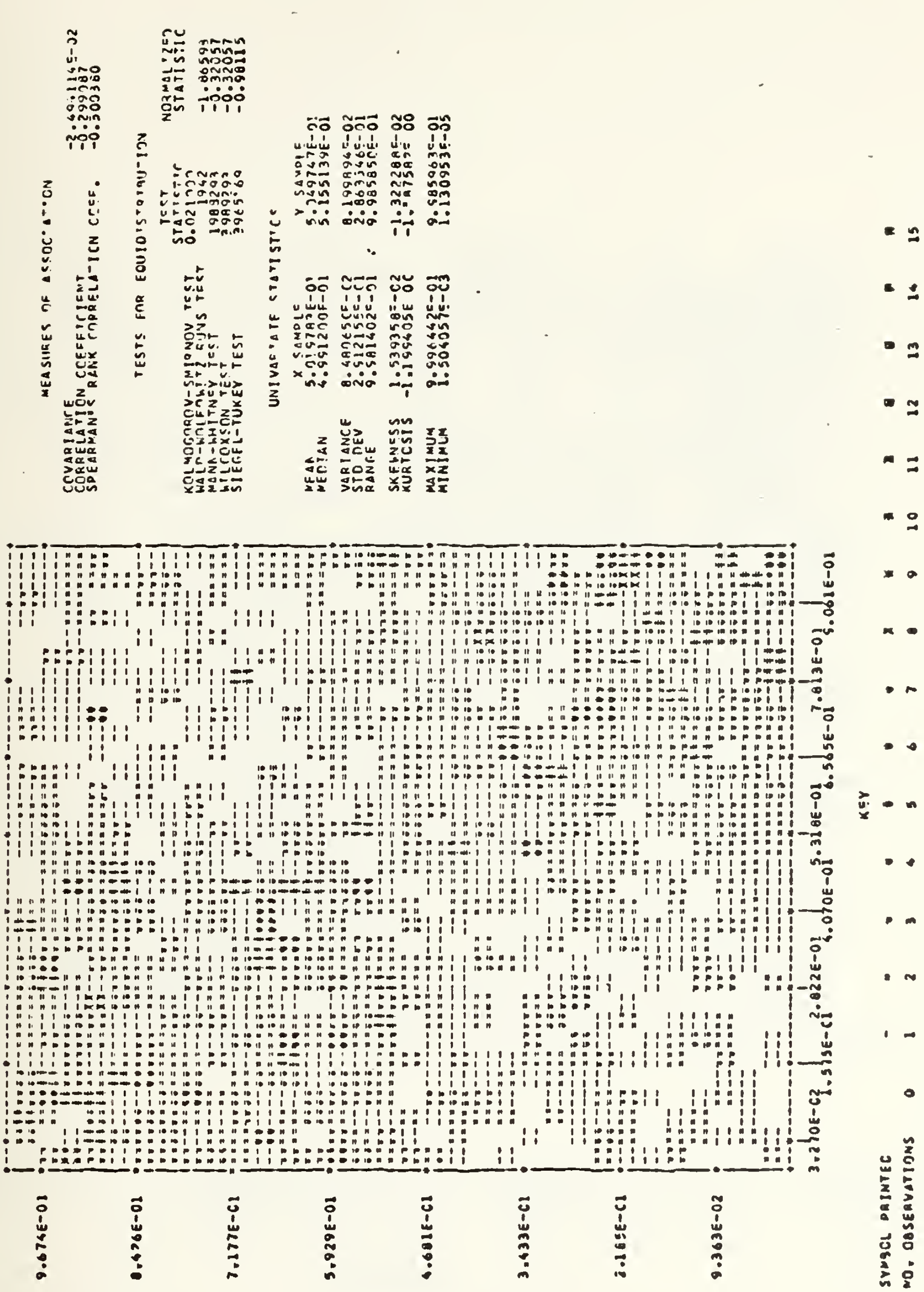


Figure IV-c4. Bivariate histogram for sample of size 2000 from mixture-truncation bivariate uniform with $\rho = -0.3$. Here x_0 has triangular distribution between the lower and the upper bounds.

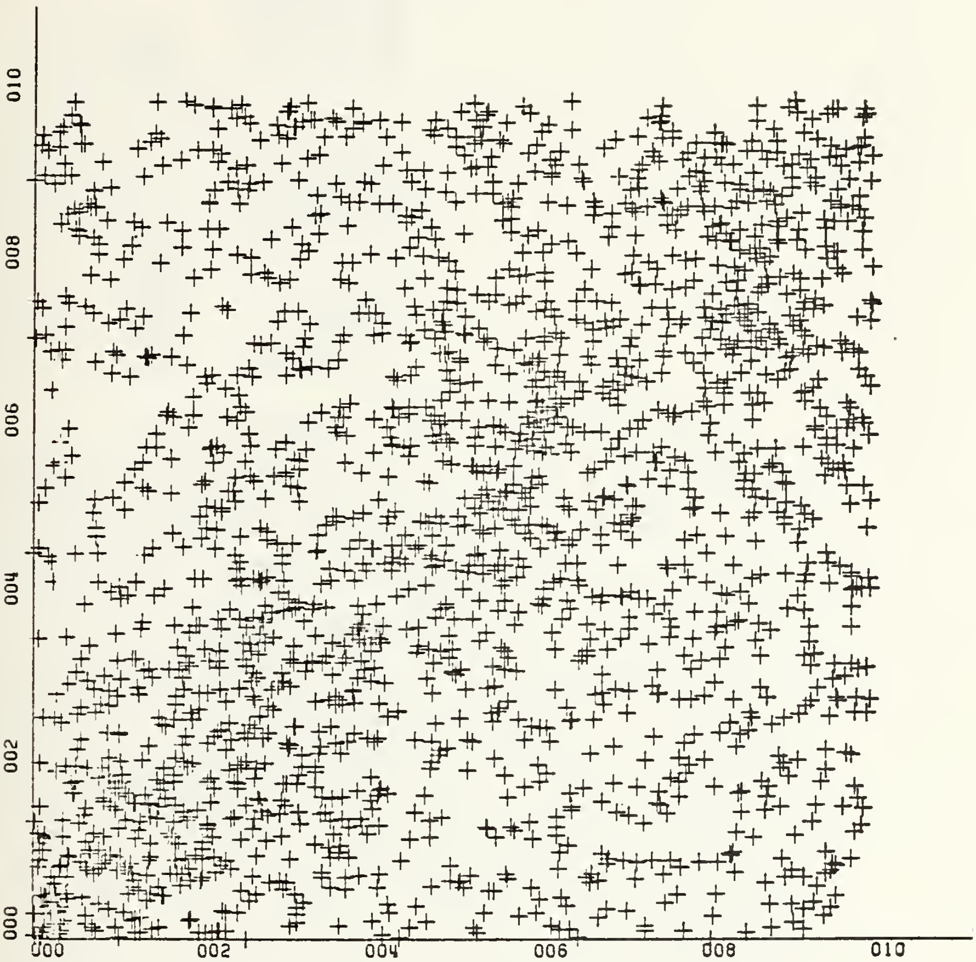
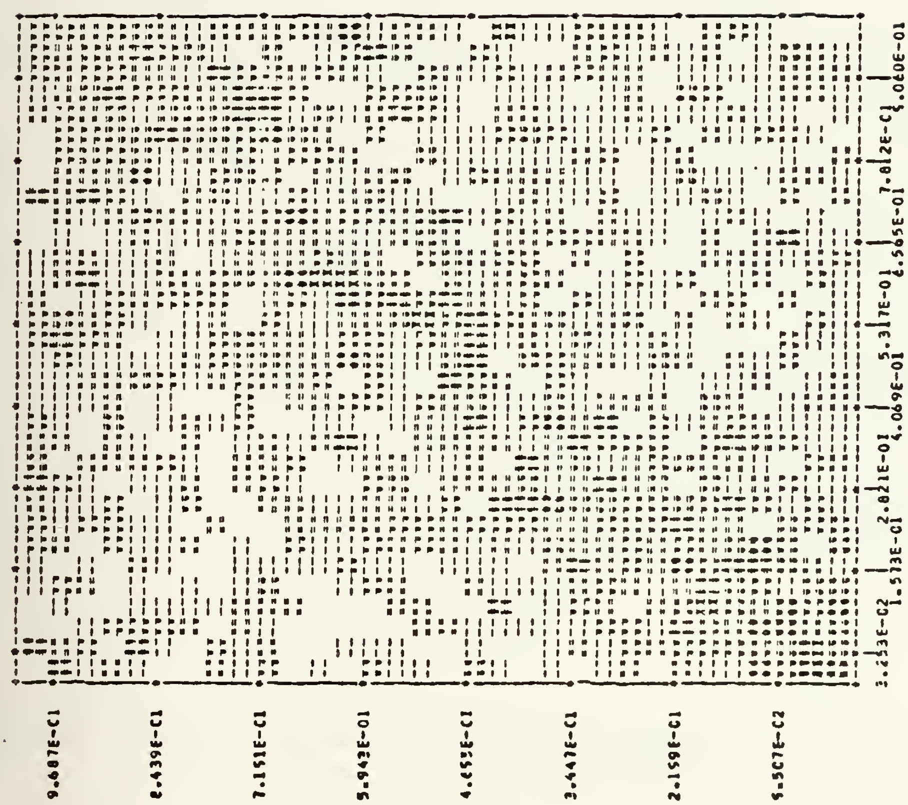


Figure IV-d1. Scatter plot for sample of size 2000 from Lawrance-Lewis's method with $\rho = 0.3$ and

$$\alpha = \frac{(2+\beta)}{3+(2+\beta)\rho}\left(\frac{1}{\beta}+1\right)$$

and β is uniform over $[\frac{2\rho}{3-\rho}, 1]$.

BIVARIATE SAMPLE SUMMARY SAMPLE SIZE = 2000



KEY

SYMBOL PRINTED		-	=	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
NO. OBSERVATIONS	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

MEASURES OF ASSOCIATION

COVARIANCE	2.191508E-02
CORRELATION COEFFICIENT	0.263363
SPEARMAN'S RANK CORRELATION COEF.	0.263660

TESTS FOR EQUIDISTRIBUTION

	TEST STATISTIC	NORMALIZED STATISTIC
KOLMOGOROV-SMIRNOV TEST	0.023033	
WALD-WOLFOWITZ RUNS TEST	2006	0.15813
MANN-WHITNEY TEST	2024358	0.66659
WILCOXON TEST	4025258	0.66659
SIEGEL-TUKEY TEST	4006003	0.13700

UNIVARIATE STATISTICS

	X SAMPLE	Y SAMPLE
MEAN	5.032177E-01	4.570405E-01
MEDIAN	5.052371E-01	4.591197E-01
VARIANCE	8.306557E-02	8.735942E-02
STD DEV	2.882110E-01	2.687202E-01
RANGE	9.562905E-01	9.584771E-01
SKEWNESS	4.961876E-06	6.142035E-03
KURTOSIS	-1.197403E 00	-1.196585E 00
MAXIMUM	9.556211E-01	9.999356E-01
MINIMUM	1.330197E-02	1.459457E-03

Figure IV-d2. Bivariate histogram for sample of size 2000 from Lawrance-Lewis's method with $\rho = 0.3$ and

$$\alpha = \frac{(2+\beta)}{3+(2+\beta)\rho}\left(\frac{1}{\beta}+1\right)$$

and β is uniform over $\left[\frac{2\rho}{3-\rho}, 1\right]$.

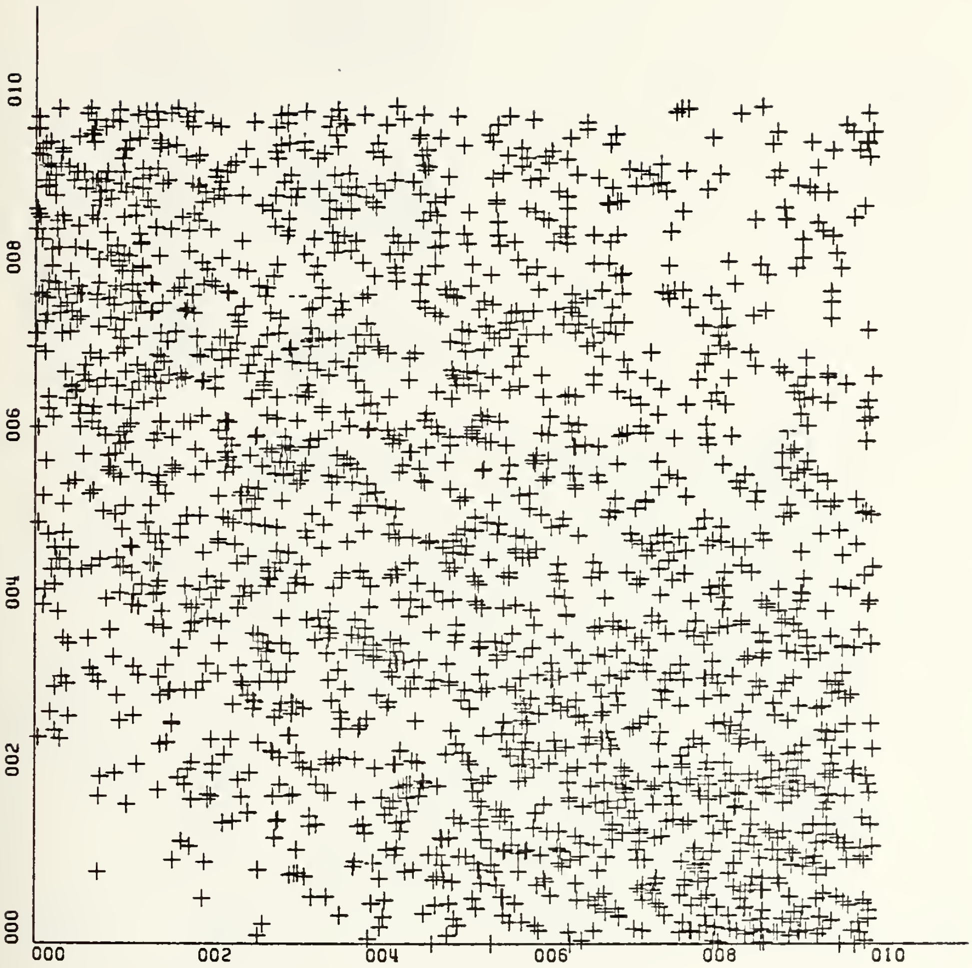


Figure IV-el. Scatter plot for sample of size 2000 from Gaver's transformation with p = 0.6 .

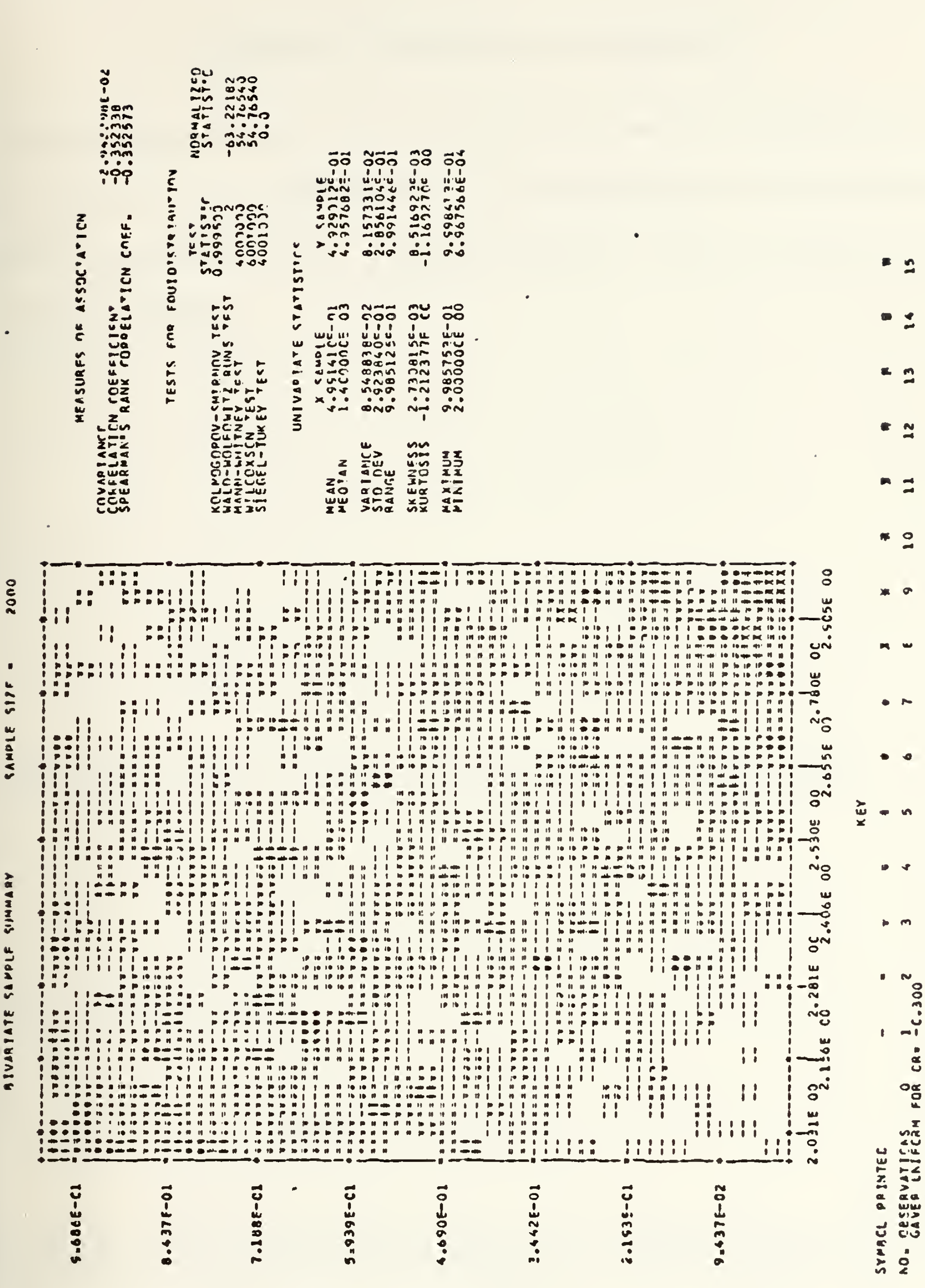


Figure IV-e2. Bivariate histogram for sample of size 2000 from Gaver's transformation with p = 0.6.

V. BIVARIATE EXPONENTIAL GENERATOR

Another probability distribution of major interest in simulations is the exponential. The cumulative distribution function and probability density function for the exponential are, respectively,

$$F(x) = 1 - e^{-\lambda x} \qquad x \geq 0$$

and

$$f(x) = \lambda e^{-\lambda x} \qquad x \geq 0$$

The expected value of the exponential distribution is

$$E[X] = 1/\lambda$$

and the variance is

$$VAR[X] = 1/\lambda^2 .$$

The problem of generating exponential deviates reduces to one of generating "unit" exponentials, i.e., those with $\lambda = 1.0$, and then multiplying the result by whichever $1/\lambda$ is necessary to give the desired distribution. That is, if the random variable E has the exponential ($\lambda = 1$) distribution then,

$$X = E * \frac{1}{n}$$

has the exponential ($\lambda = n$) distribution. In this section, we considered only unit exponentials as marginal distributions for bivariate pairs.

A. DETERMINATION OF PARAMETERS IN THE EXPONENTIAL MIXTURE-TRUNCATION METHOD

Because X_1 is an exponential ($\lambda = 1$) truncated to the left of x_o and X_2 is also exponential ($\lambda = 1$) truncated to the right of x_o, we have

$$E[X_1] = \mu_1 = \int_0^{x_o} x \, d\frac{F(x)}{F(x_o)}$$

$$= 1 - \frac{\pi_2}{\pi_1} x_o;$$

$$VAR[X_1] = \sigma_1^2 = \int_0^{x_o} x^2 \, d\frac{F(x)}{F(x_o)} - \mu_1^2$$

$$= 1 - \frac{\pi_2}{\pi_1^2} x_o^2;$$

$$E[X_2] = \mu_2 = \int_{x_o}^{\infty} x \, d\frac{F(x) - F(x_o)}{1 - F(x_o)}$$

$$= 1 + x_o;$$

$$\mathrm{VAR}[X_2] = \int_{x_o}^{\infty} x^2 \, d \, \frac{F(x) - F(x_o)}{1 - F(x_o)} - \mu_2^{\,2}$$

$$= 1;$$

and from definition,

$$\pi_1 = \frac{1 - \alpha_2}{1 - \alpha_1 + 1 - \alpha_2}$$

$$= F(x_o)$$

$$= 1 - e^{-x_o} , \qquad \text{(V-A-1a)}$$

$$\pi_2 = \frac{1 - \alpha_1}{1 - \alpha_1 + 1 - \alpha_2}$$

$$= 1 - F(x_o)$$

$$= e^{-x_o} . \qquad \text{(V-A-1b)}$$

If we use these formulas in Theorem 2 in Section III, we get

$$M = \frac{\pi_2}{\pi_1} x_o^{\,2} \qquad \text{(V-A-2)}$$

$$\beta = \frac{1}{\pi_2}(\alpha_1 - \pi_1) \qquad \text{(V-A-3)}$$

and

$$\rho = \frac{\alpha_1 - \pi_1}{\pi_1} x_o^2 \qquad \text{(V-A-4)}$$

For given correlation coefficient ρ, we can compute α_1 as a function of x_o from the formula (V-A-4) as

$$\alpha_1 = \frac{\rho\pi_1}{x_o^2} + \pi_1$$

$$= (1 + \frac{\rho}{x_o^2})(1 - e^{-x_o})$$

and we know that $\alpha_2 = 1 - \frac{\pi_1}{\pi_2}(1 - \alpha_1)$ from the formulas (V-A-1a) and (V-A-1b). From this,

$$\alpha_2 = 1 - \frac{\pi_1}{\pi_2}(1 - \alpha_1)$$

$$= e^{-x_o} + e^{x_o}(1 - e^{-x_o})^2 \frac{\rho}{x_o^2}$$

These α_1 and α_2 are probabilities, so they have to be greater than or equal to zero and less than or equal to one;

$$0 \leq (1 - e^{-x_o})(1 + \frac{\rho}{x_o^2}) \leq 1 \qquad \text{(V-A-5a)}$$

$$0 \leq e^{-x_o} + e^{x_o}(1 - e^{-x_o})^2 \frac{\rho}{x_o^2} \leq 1 \qquad \text{(V-A-5b)}$$

From these two inequality equations, (V-A-5a) and (V-A-5b), we can find the x_o ranges for given correlation coefficient ρ. To solve these equations, we can divide into two cases,

one for positive correlation case and the other for negative correlation case. If correlation coefficient ρ is positive, then both equations are always positive. Thus we only need to find the x_o ranges which makes (V-A-5a) and (V-A-5b) less than 1. From the equation (V-A-5a), for the α_1 case,

$$\alpha_1 = (1 - e^{-x_o})(1 + \frac{\rho}{x_o^2}) \leq 1$$

becomes

$$\rho(e^{x_o} - 1) \leq x_o^2$$

and let

$$y_1 = x_o^2, \qquad y_2 = \rho(e^{x_o} - 1).$$

Because of the first derivatives of y_1 and y_2 are always positive, we know that these two functions are monotone increasing functions. Thus we can find x_o ranges which satisfy $y_1 \geq y_2$ by the Newton Raphson method. When using the Newton Raphson method, let $y = y_1 - y_2 = 0$ and find an approximate solution, by approximating exponential series, which we can use as a starting point. That is,

$$y = y_1 - y_2 = x_o^2 - \rho(1 + x_o + \frac{x_o^2}{2!} + \frac{x_o^3}{3!} - 1)$$

$$= \frac{\rho}{6} x_o^2 - (1 - \frac{\rho}{2}) x_o + \rho = 0$$

Then

$$x_o = \frac{(1 - \frac{\rho}{2}) \pm (1 - \rho - \frac{5}{12} \rho^2)^{1/2}}{1/3 \rho}$$

Starting with this approximate value in the Newton Raphson method, we can find x_o range, say $[x_{\ell_1}, x_{u_1}]$, which satisfies $0 \leq \alpha_1 \leq 1$. And for the α_2 case,

$$\alpha_2 = e^{-x_o} + e^{-x_o}(1 - e^{-x_o})^2 \frac{\rho}{x_o^2} \leq 1$$

becomes

$$\rho(e^{x_o} - 1) \leq x_o^2$$

This result is exactly the same as the α_1 case, that is, at the same range α_1 and α_2 satisfy constraints $\alpha_1 \leq 1$ and $\alpha_2 \leq 1$. Thus we can use x_{ℓ_1} and x_{u_1} as the lower bound of x_o, x_ℓ, and the upper bound of x_o, x_u. If correlation coefficient ρ is negative, then the equations (V-A-5a) and (V-A-5b) are always less than 1. Therefore we need to consider only one constraint which makes $\alpha_1 \geq 0$, $\alpha_2 \geq 0$. From the α_1 equation (V-A-5a), solve the inequality equation

$$0 \leq \alpha_1 = (1 - e^{-x_o})(1 + \frac{\rho}{x_o^2})$$

Since $1 - e^{-x_o}$ is always positive, we see that to satisfy the inequality $1 + \frac{\rho}{x_o^2}$ should be positive, i.e.,

$$\frac{\rho}{x_o^2} \geq -1$$

equivalently, we have

$$x_o \geq \sqrt{-\rho}$$

In the α_2 case, from equation (V-A-5b),

$$0 \leq \alpha_2 = e^{-x_o} + e^{x_o}(1 - e^{-x_o})^2 \frac{\rho}{x_o^2}$$

or, equivalently, we have

$$x_o^2 \geq -\rho(e^{x_o} - 1)^2$$

As in the positive correlation case, we can find a starting point by approximation to solve this equation by Newton Raphson. The result comes out as

$$x_s = (\sqrt{-\rho} + \rho)/(-\tfrac{\rho}{2})$$

with this starting point we find another bound of x_o which satisfies $0 \leq \alpha_2$. This becomes the upper bound of x_o, x_u, and from the α_1 case, we have a constraint $x_o \geq \sqrt{-\rho}$ which

becomes the lower bound of x_o, i.e., $x_\ell = \sqrt{-\rho}$. The lowest and highest correlations available for bivariate exponential pairs in mixture-truncation method are approximately -0.480 and 0.647 respectively. By comparison note that the most negative correlation available for bivariate exponential pairs with identical fixed marginals is $1 - \frac{\pi^2}{6}$ [Moran (1967)]. Gaver's (1972) negatively correlated pair has correlations in the range [-0.5,0]. The table (V-1) shows the lower and upper bound of x_o in the mixture-truncation method with identical marginal exponential and given correlation.

Table V-1: The lower bound and upper bound of x_o in the mixture-truncation method with identical exponential marginal distributions

x_o range for each correlation

ρ	x_L	x_u	ρ	x_L	x_u
0.1	0.106	5.832	-0.1	0.317	1.984
0.2	0.225	4.723	-0.2	0.448	1.439
0.3	0.362	3.990	-0.3	0.548	1.103
0.4	0.527	3.395	-0.4	0.633	0.855
0.5	0.741	2.842	-0.45	0.671	0.751
0.6	1.082	2.223			

B. GENERATING PROCEDURE

The generating procedure of bivariate exponential random vector is almost the same as in the uniform case. As in the uniform case, we develop three procedures which are the FXO method, the UXO method and the TXO method. As mentioned

in Section III, all of these methods are exactly the same except in how x_o is chosen from the x_o range $[x_\ell, x_u]$. The FXO method chooses x_o as a fixed point between x_ℓ and x_u. This procedure makes some discontinuity at the truncation point. In this respect, this method is defective.

The UXO and TXO methods are the same except we use different distributions for x_o. In the UXO method we use the uniform distribution and the triangular distribution for the TXO method. These methods have more smooth distribution than the FXO method.

<u>Exponential Mixture-Truncation Method</u>

1. (Initialization)

   i) For given $-0.48 < \rho < 0.64$, find x_ℓ and x_u

2. Define truncation point x_o

   * FXO method

     i) $x_o = \frac{1}{2}(x_\ell + x_u)$

   * UXO method

     i) Generate a uniform [0,1] random variable U_1

     ii) $x_o = x_\ell + (x_u - x_\ell) * U_1$

   * TXO method

     i) Generate two uniform [0,1] random variables V_1, V_2

     ii) $x_o = x_\ell + x_1 + x_2$

     where

$$x_m = (x_\ell + x_u)/2$$

$$x_1 = (x_m - x_\ell) * V_1$$

$$x_2 = (x_u - x_m) * V_2$$

3. Compute parameter values

$$\pi_1 = F(x_o) = 1 - e^{-x_o}$$

$$\pi_2 = 1 - \pi_1$$

$$\alpha_1 = \pi_1 (1 + \frac{\rho}{x_o^2})$$

$$\alpha_2 = 1 - \frac{\pi_1}{\pi_2}(1 - \alpha_1)$$

4. Choose type for Y

   i) Generate a uniform [0,1] random variable U

   ii) If $U \leq \pi_1$, go to 9

5. Y is an X_2

   i) Generate an exponential random variable E_1

   ii) If $E_1 > x_o$, set $Y \leftarrow E_1$ and go to 6

   iii) Otherwise, return to 5.i)

6. Choose type for Z

   i) Set $U \leftarrow ((U - \pi_1)/(1 - \pi_1))$

ii) If $U \le 1 - \alpha_2$, go to 8

7. Z is an X_2

    i) Generate an exponential random variable E_2

    ii) If $E_2 > x_o$, set $Z \leftarrow E_2$ and go to 11

    iii) Otherwise return to 7.i)

8. Z is an X_1

    i) Generate an exponential random variable E_2

    ii) If $E_2 \le x_o$, set $Z \leftarrow E_2$ and go to 11

    iii) Otherwise return to 8.i)

9. Y is an X_1

    i) Generate an exponential random variable E_1

    ii) If $E_1 \le x_o$, set $Y \leftarrow E_1$ and go to 10

    iii) Otherwise return to 9.i)

10. Choose type for Z

    i) Set $U \leftarrow U/\pi_1$

    ii) If $U \le \alpha_1$, go to 8

    iii) Otherwise go to 7

11. Deliver (Y,Z) and go to 4 for the FXO method, or go to 2 for the UXO and TXO methods until a sufficient number of random vectors are obtained.

For the exponential case it is possible to give a more efficient algorithm in which X_1 and X_2 are generated exactly. The algorithm is as follows.

Efficient Exponential Mixture-Truncation Method

1. (Initialization)

   i) For given $-0.48 < \rho < 0.64$, find x_ℓ and x_u

2. Define the truncation point x_o

   * FXO method

     $$x_o = \frac{1}{2}(x_\ell + x_u)$$

   * UXO method

     i) Generate a uniform [0,1] random variable U_1

     ii) $x_o = x_\ell + (x_u - x_\ell) * U_1$

   * TXO method

     i) Generate two uniform [0,1] random variables V_1, V_2

     ii) $x_o = x_\ell + x_1 + x_2$

     where

     $$x_m = (x_\ell + x_u)/2$$

     $$x_1 = (x_m - x_\ell) * V_1$$

     $$x_2 = (x_u - x_m) * V_2$$

3. Compute parameter values

$$\pi_1 = F(x_o) = 1 - e^{-x_o}$$

$$\pi_2 = 1 - \pi_1$$

$$\alpha_1 = \pi_1(1 + \frac{\rho}{x_o^2})$$

$$\alpha_2 = 1 - \frac{\pi_1}{\pi_2}(1 - \alpha_1)$$

4. Choose type for Y

   i) Generate a uniform [0,1] random variable U

   ii) If $U \leq \pi_1$, go to 9

5. Y is an X_2

   i) Generate an exponential random variable E_1

   ii) Set $Y \leftarrow x_o + E_1$

6. Choose type for Z

   i) Set $U \leftarrow ((U - \pi_1)/(1 - \pi_1))$

   ii) If $U \leq 1 - \alpha_2$, go to 8

7. Z is an X_2

   i) Generate an exponential random variable E_2

   ii) Set $Z \leftarrow x_o + E_2$ and go to 11

8. Z is an X_1

   i) Generate a uniform [0,1] random variable W_2

   ii) Set $Z \leftarrow -\ln(1.0 - W_2*\pi_1)$ and go to 11

9. Y is an X_1

   i) Generate a uniform [0,1] random variable W_1

ii) Set $Z \leftarrow -\ln(1.0 - W_1 * \pi_1)$

10. Choose type for Z

i) Set $U \leftarrow U/\pi_1$

ii) If $U \leq \alpha_1$, go to 8

iii) Otherwise, go to 7

11. Deliver (Y,Z) and go to 4 for the FXO method, or go to 2 for the UXO and TXO methods until a sufficient number of random vectors are obtained.

Note that to compute x_ℓ and x_u in step 1 of both algorithms we use subroutine BOUND which is listed in the appendix and the program with effective algorithm also. The scatter plot and bivariate histogram of resulting random vectors are shown in Section V-D.

C. REGRESSION OF Z ON Y FOR GIVEN ρ

Schmeiser (1979) has used the regression of Z on $Y = y$ to fix the parameters in his bivariate gamma distribution. Consequently we investigate this for the mixture-truncation method case. The regression is different depending on whether $Y \leq x_o$ or $Y > x_o$. We consider two cases here, one for fixed x_o and the other for x_o having uniform distribution. For fixed x_o, we have

$$E[Z|Y=y, y \leq x_o] = \alpha_1 E[x_1] + (1-\alpha_1)E[x_2]$$

$$= 1 + x_o - \alpha_1 x_o\left(1 + \frac{\pi_2}{\pi_1}\right)$$

Substituting the value for α_1,

$$\alpha_1 = \pi_1(1 + \frac{\rho}{x_o^2})$$

then we have

$$E[Z|Y=y,y \leq x_o] = 1 + x_o - x_o(1 + \frac{\rho}{x_o^2})$$

$$= 1 - \frac{\rho}{x_o}$$

And if $y > x_o$, then

$$E[Z|Y=y,y > x_o] = (1-\alpha_2)E[x_1] + \alpha_2 E[x_2]$$

$$= 1 - \frac{\pi_2}{\pi_1} x_o + \frac{x_o}{\pi_1} \alpha_2$$

Substituting the value for α_2,

$$\alpha_2 = 1 - \frac{\pi_1}{\pi_2}(1-\alpha_1)$$

$$= 1 - \pi_1(1 - \frac{\pi_1}{\pi_2} \frac{\rho}{x_o^2})$$

then we have

$$E[Z|Y=y,y > x_o] = 1 + \frac{\pi_1}{\pi_2} \frac{\rho}{x_o}$$

Thus the regression is constant over $(0,x_o)$ and changes for $y \geq x_o$. This is not surprising in light of the joint distribution given in Section IV-D. For uniformly distributed x_o, the computation is different for different ranges of Y. If $y \leq x_\ell$, then we have

$$E[Z|Y=y] = \int_{x_\ell}^{x_u} E[Z|Y=y,\ X=x_o,\ y \leq x_o]\ f(x_o)\ dx_o$$

$$= \int_{x_\ell}^{x_u} (1 - \frac{\rho}{x_o^2})\ e^{-x_o}\ dx_o$$

$$= -\ e^{-x_u} + e^{-x_\ell} - \rho \int_{x_\ell}^{x_u} \frac{e^{-x_o}}{x_o^2}\ dx_o$$

$$= e^{-x_\ell} - e^{-x_u} + \rho(\frac{1}{x_u}\ e^{-x_u} - \frac{1}{x_\ell}\ e^{-x_\ell}$$

$$+ \ln \frac{x_u}{x_\ell} - x_u + x_\ell)$$

If

$$x_\ell \leq y \leq x_u$$

then we have

$$E[Z|Y=y] = \int_{x_L}^{y} E[Z|Y=y,\ X=x_o,\ y \geq x_o] f(x_o)\ dx_o$$

$$+ \int_{y}^{x_u} E[Z|Y=y,\ X=x_o,\ y \leq x_o] f(x_o)\ dx_o$$

$$= 2e^{-y} + 2\rho y - \rho(x_\ell + x_u) + \rho\left(\frac{1}{x_u} e^{-x_u} - \frac{1}{y}e^{-y}\right)$$

$$+ \rho \ln \frac{x_u}{y}$$

If $y > x_u$, then we have

$$E[Z|Y=y] = \int_{x_\ell}^{x_u} E[Z|Y=y,\ X=x_o,\ y > x_o] f(x_o)\ dx_o$$

$$= \int_{x_\ell}^{x_u} \left(1 + \frac{\pi_1}{\pi_2}\frac{\rho}{x_o}\right) e^{-x_o}\ dx_o$$

$$= e^{-x_\ell} - e^{-x_u} + \rho(y - x_\ell)$$

By making x_o uniformly distributed over the available range of x_o for given correlation, we can get smoother behavior for the regression function.

D. SIMULATION RESULTS

We will show here the scatter plots and bivariate historgram of the mixture-truncation bivariate exponential

random vectors with correlation $\rho = 0.3$ and $\rho = -0.3$ by the FXO, UXO and TXO methods in Figures (V-a), (V-b) and (IV-c), respectively. Also we show the results from Gaver's bivariate exponential and Marshall and Olkin's in Figures (V-d) and (V-e), respectively.

In the mixture-truncation bivariate exponential distribution, the generated random vectors by the FXO method also has discontinuity at truncation point but the other cases have relatively smooth distributions.

The computed correlation printed in the left side of the bivariate histogram is a little different from the given correlation. But this difference can be assumed as a sampling error.

To check this error we simulated 10 times with given correlation $\rho = 0.1$. From these simulations we get:

$$\bar{\rho} \;=\; \text{mean of computed correlation} \;=\; 0.092$$

$$\mathrm{VAR}[\rho] \;=\; \text{variance of computed correlation} \;=\; 0.0008$$

$$\sigma(\bar{\rho}) \;=\; \text{standard deviation of mean} \;=\; 0.009.$$

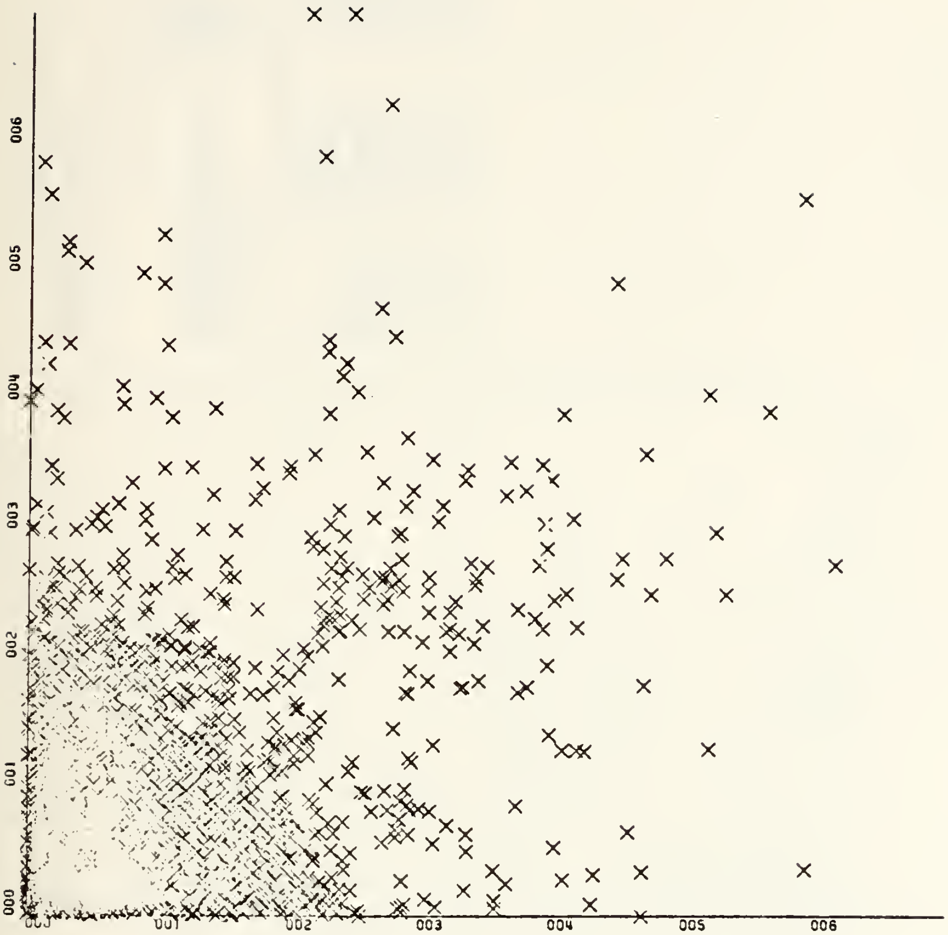


Figure V-a1. Scatter plot for sample of size 2000 from mixture-truncation bivariate exponential with $\rho = 0.3$. Here x_o is fixed at midpoint between the lower and the upper bounds.

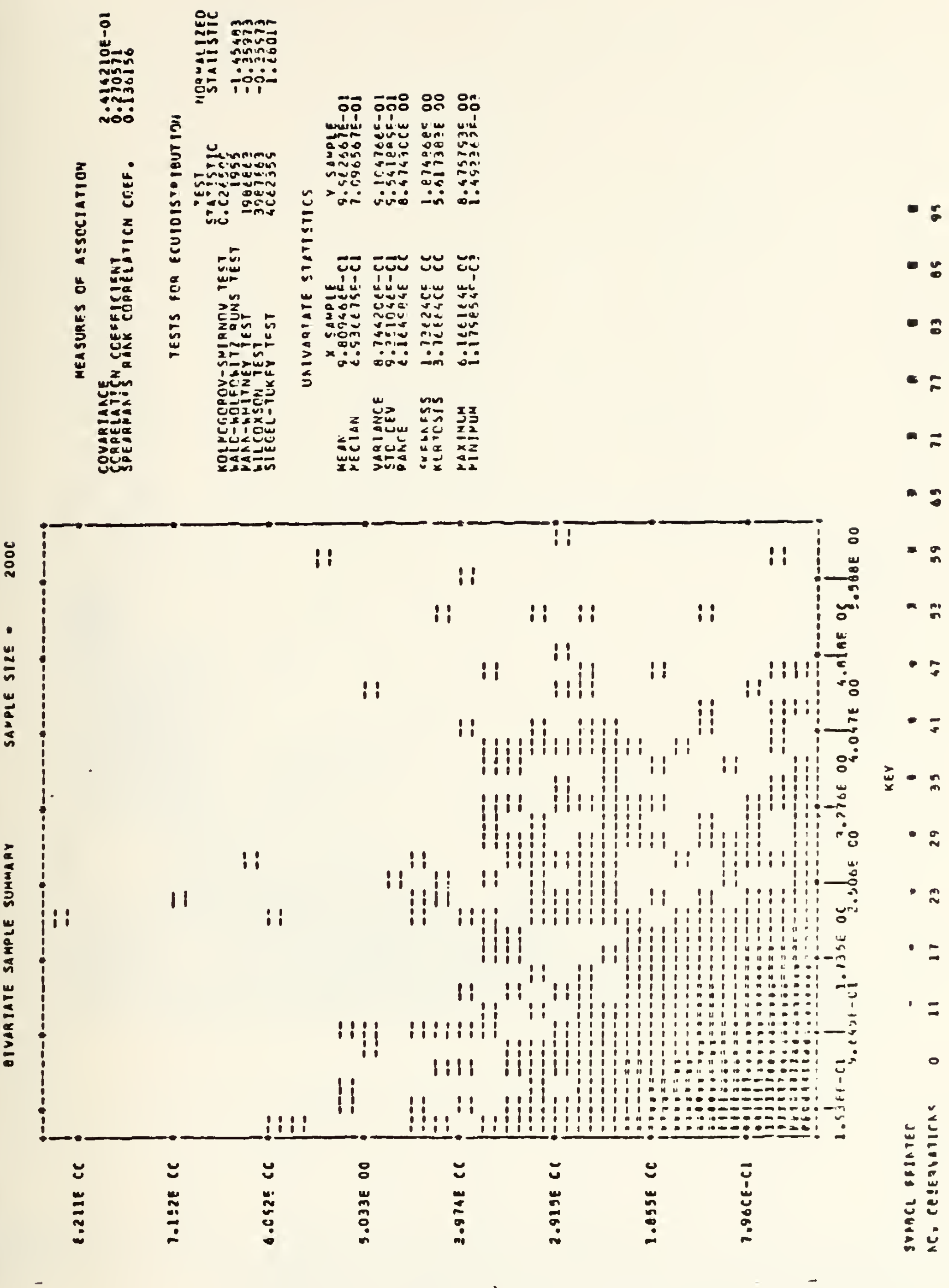


Figure V-a2. Bivariate histogram for sample of size 2000 from mixture-truncation bivariate exponential with $\rho = 0.3$. Here x_o is fixed at the midpoint between the lower and the upper bounds.

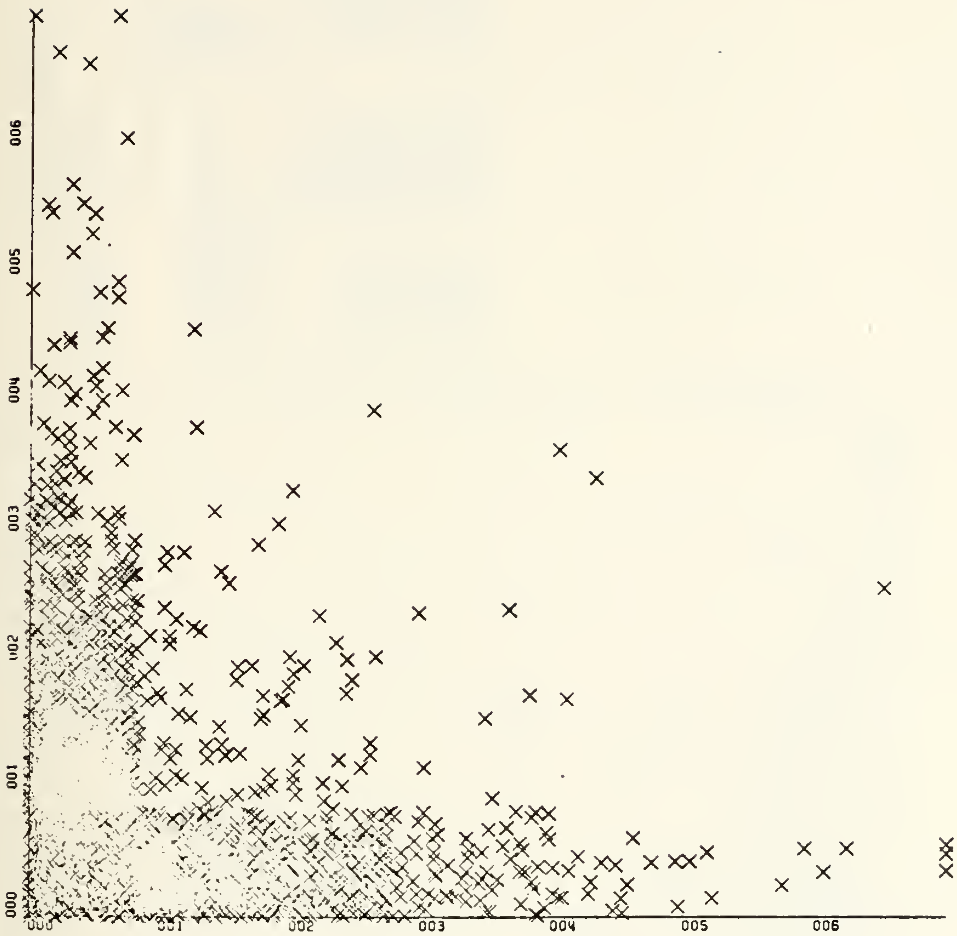


Figure V-a3. Scatter plot for sample of size 2000 from mixture-truncation bivariate exponential with $\rho = -0.3$. Here x_o is fixed at the midpoint between the lower and the upper bounds.

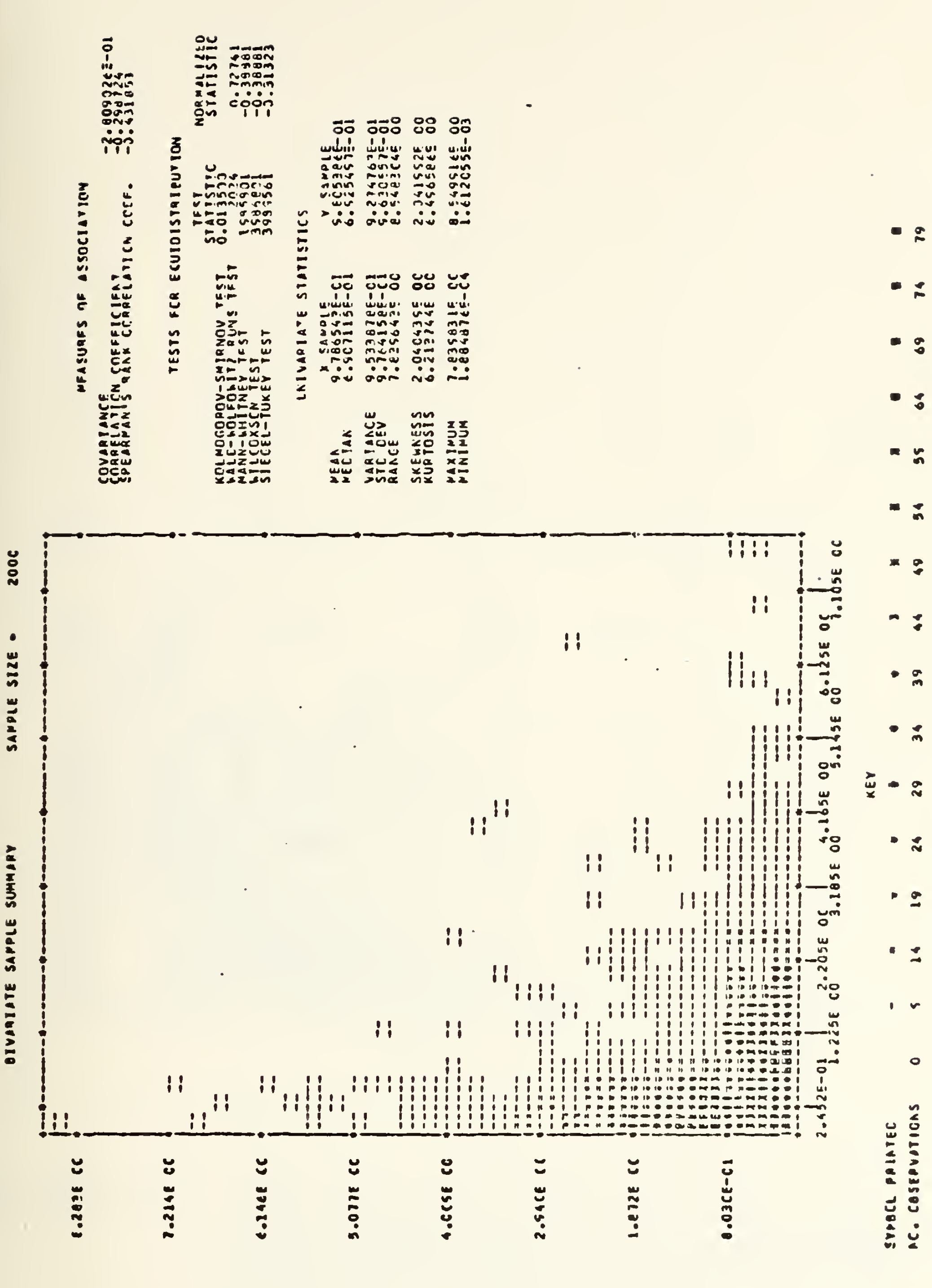


Figure V-a4. Bivariate histogram for sample of size 2000 from mixture-truncation bivariate exponential with $\rho = -0.3$. Here x_0 is fixed at the midpoint between the lower and the upper bounds.

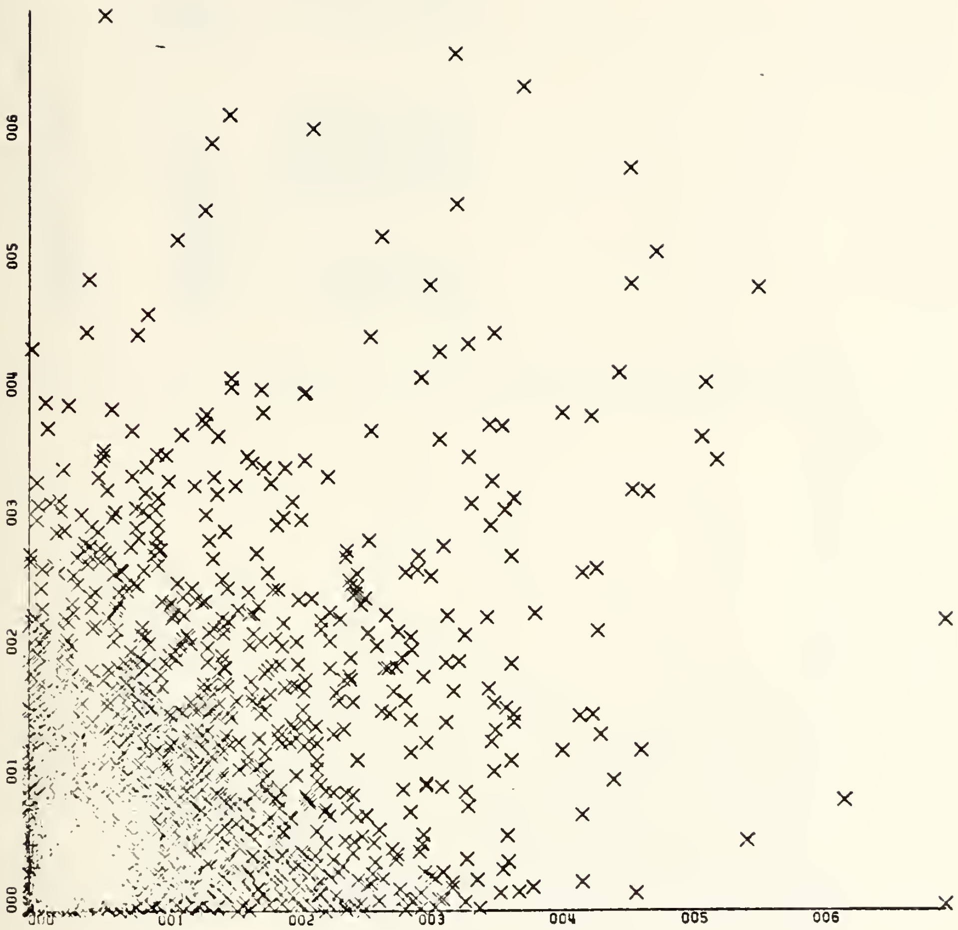


Figure V-b1. Scatter plot for sample of size 2000 from mixture-truncation bivariate exponential with $\rho = 0.3$. Here x_o is taken to be uniformly distributed between the lower and the upper bounds .

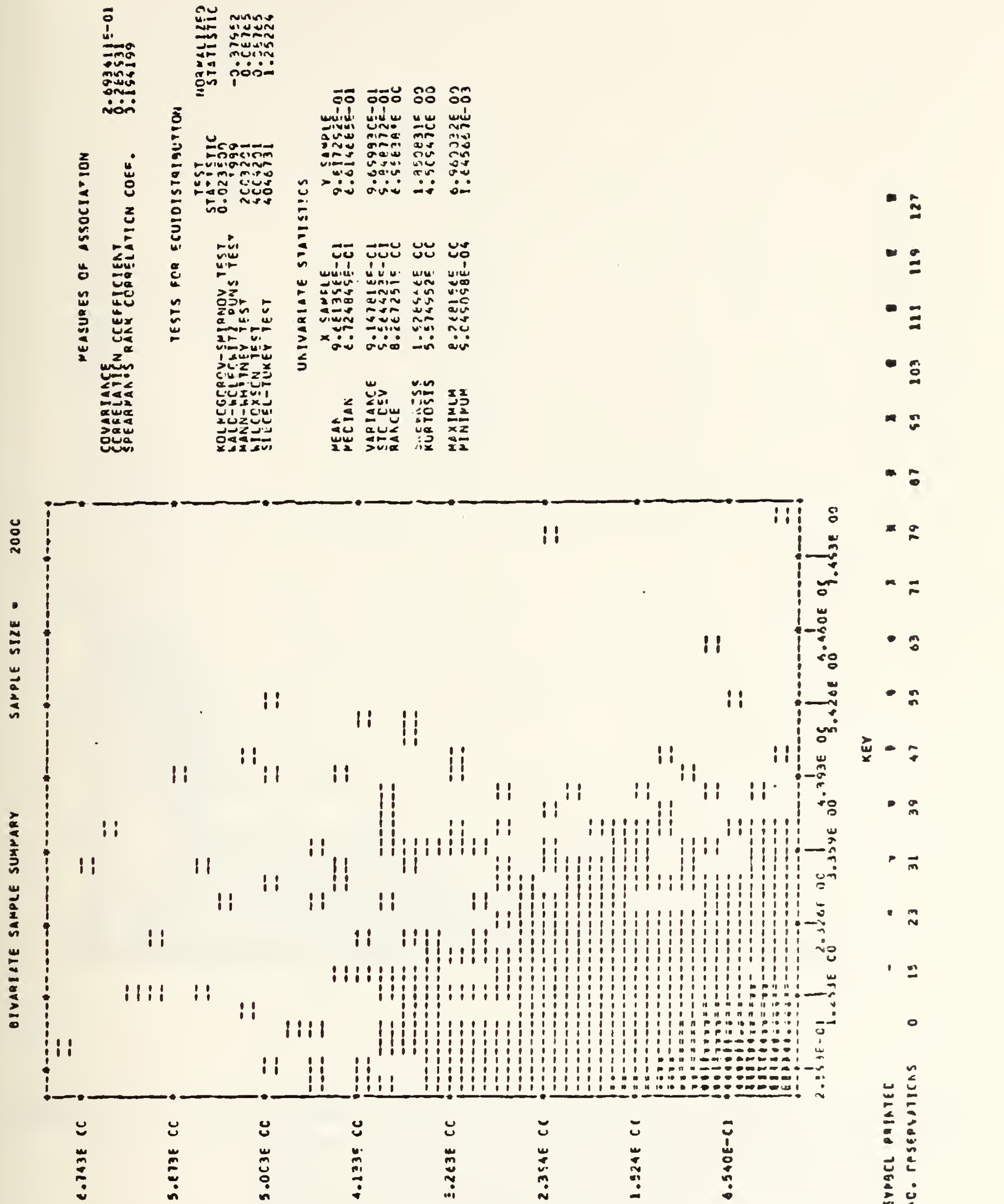


Figure V-b2. Bivariate histogram for sample of size 2000 from mixture-truncation bivariate exponential with $\rho = 0.3$. Here x_o is taken to be uniformly distributed between the lower and the upper bounds.

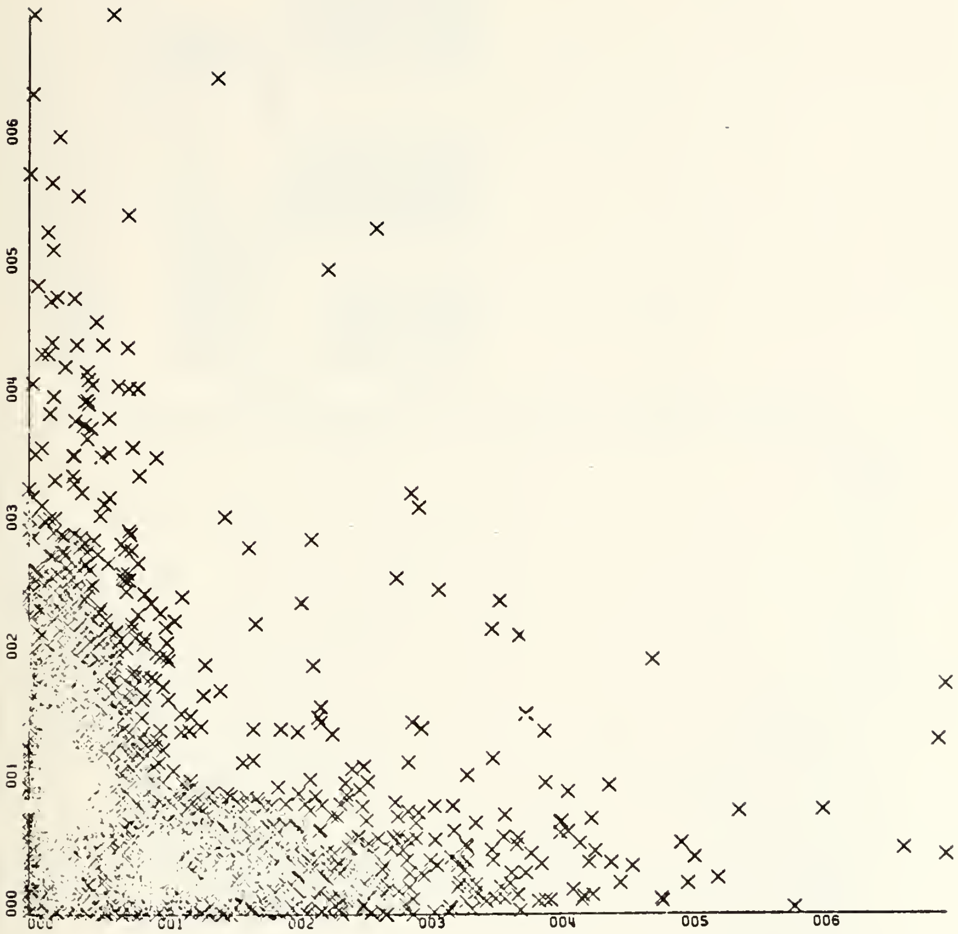


Figure V-b3. Scatter plot for sample of size 2000 from mixture-truncation bivariate exponential with $\rho = -0.3$. Here x_o is taken to be uniformly distributed between the lower and the upper bounds.

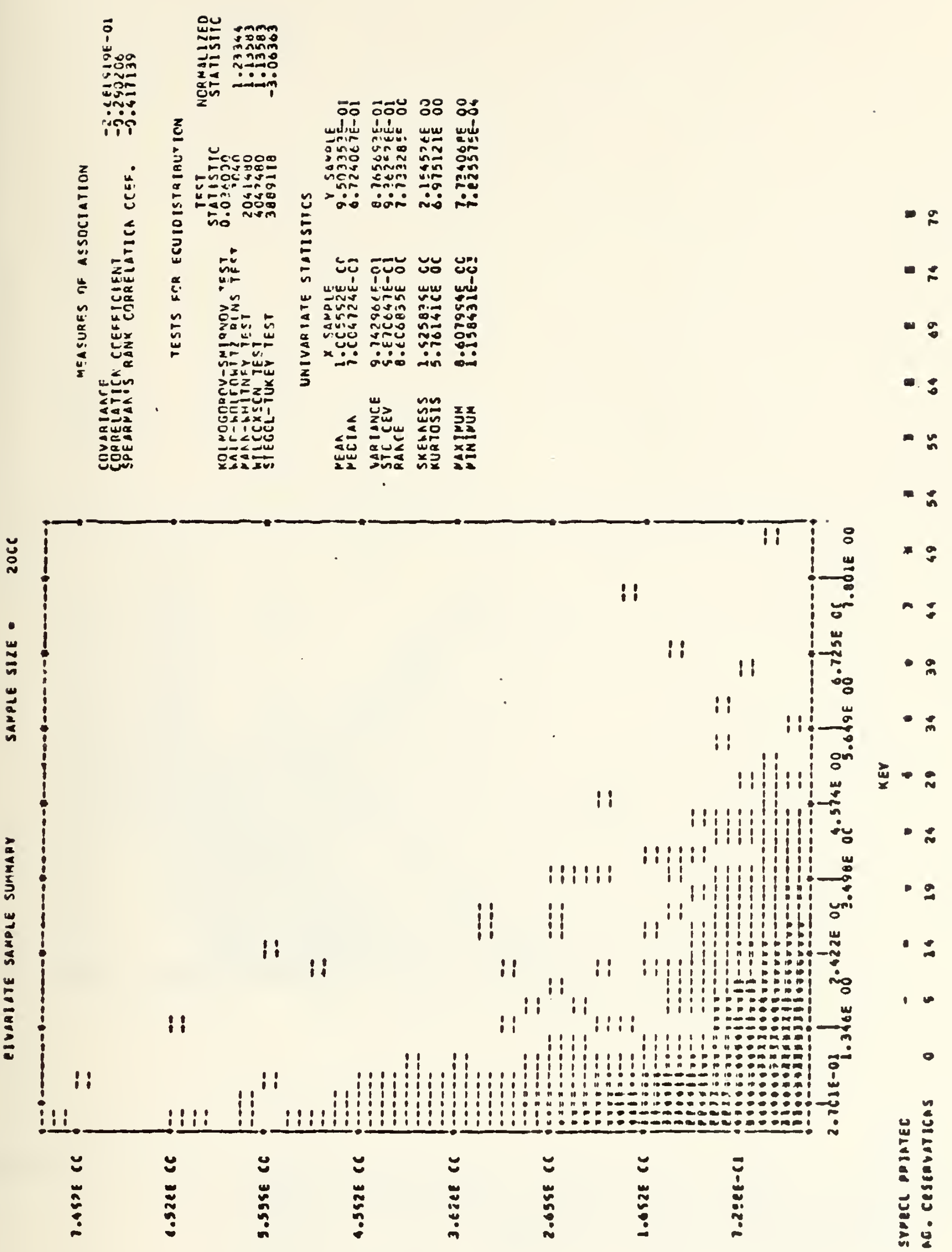


Figure V-b4. Bivariate histogram for sample of size 2000 from mixture-truncation bivariate exponential with $\rho = -0.3$. Here x_o is taken to be uniformly distributed between the lower and the upper bounds.

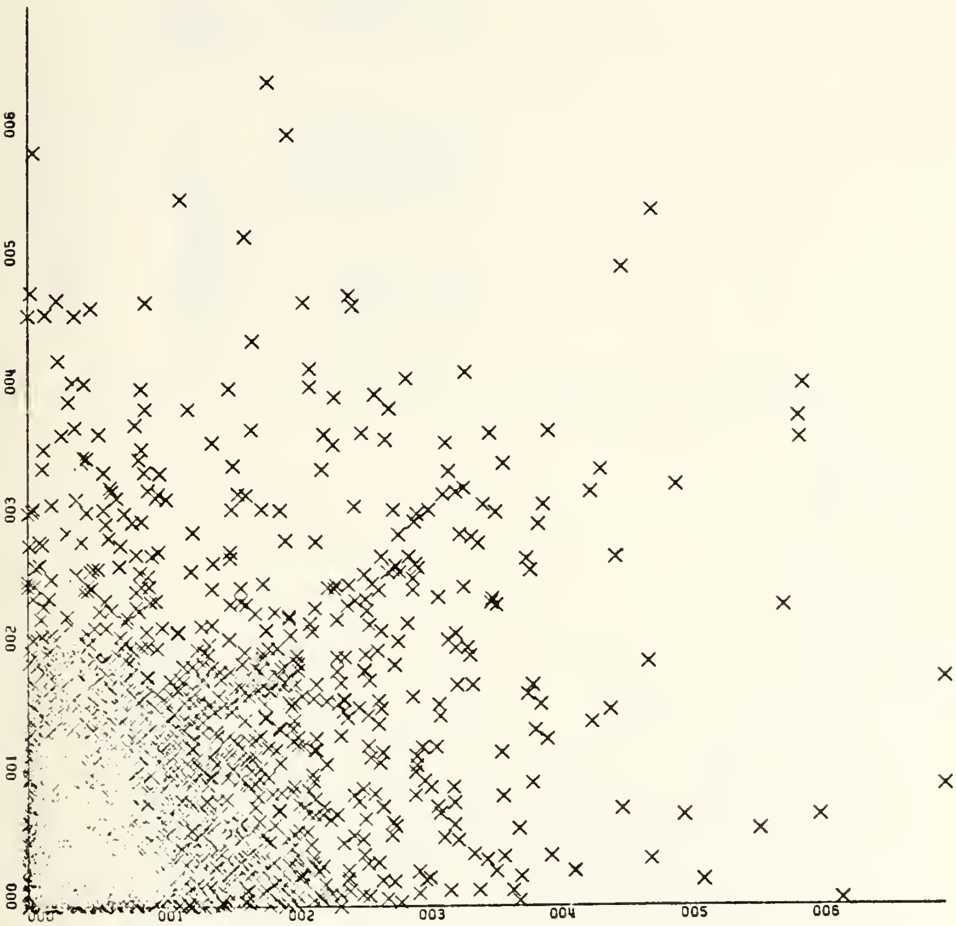


Figure V-cl. Scatter plot for sample of size 2000 from mixture-truncation bivariate exponential with $\rho = 0.3$. Here x_o has triangular distribution between the lower and the upper bounds.

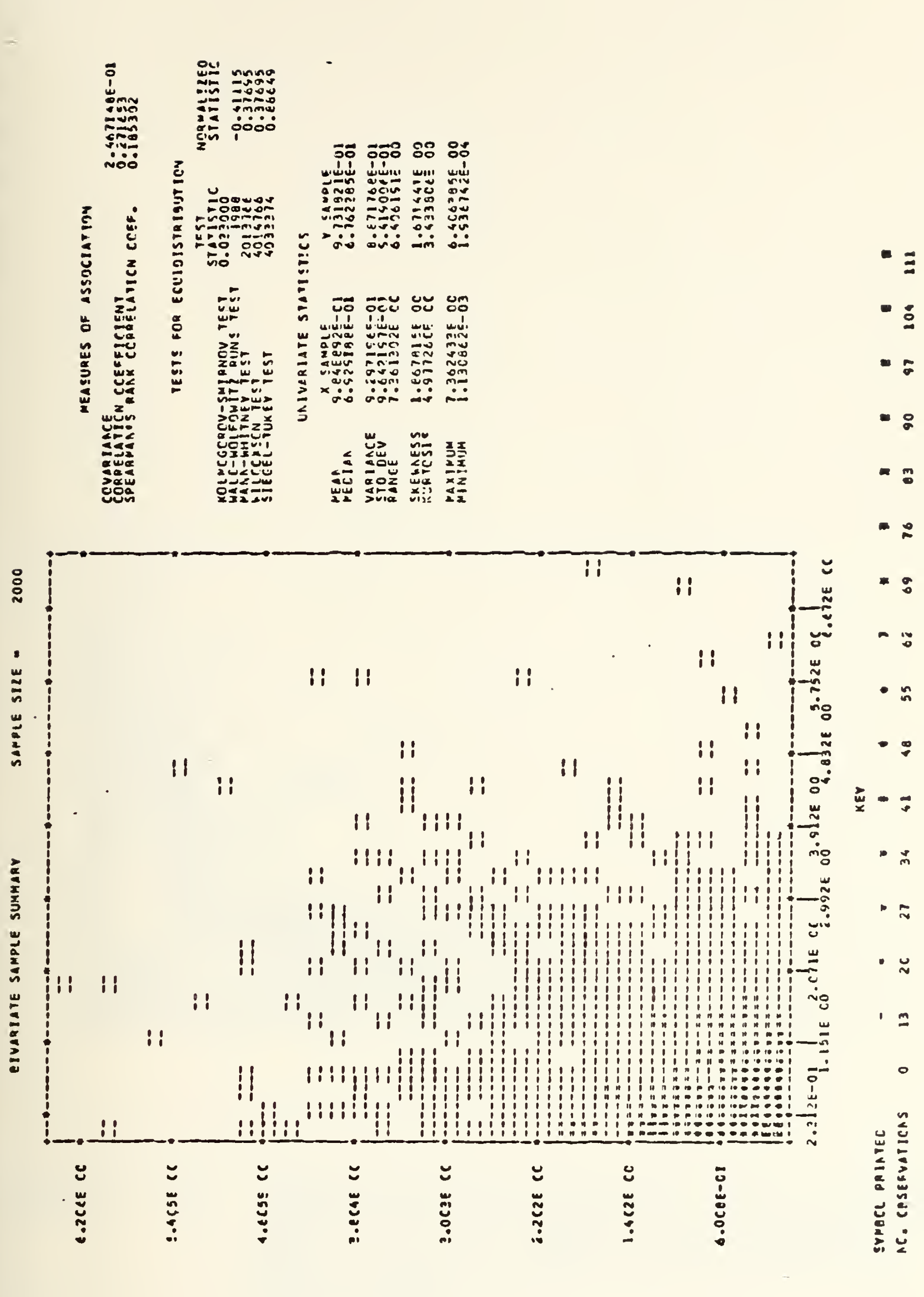


Figure V-c2. Bivariate histogram for sample of size 2000 from mixture-truncation bivariate exponential with $\rho = 0.3$. Here x_o has triangular distribution between the lower and the upper bounds.

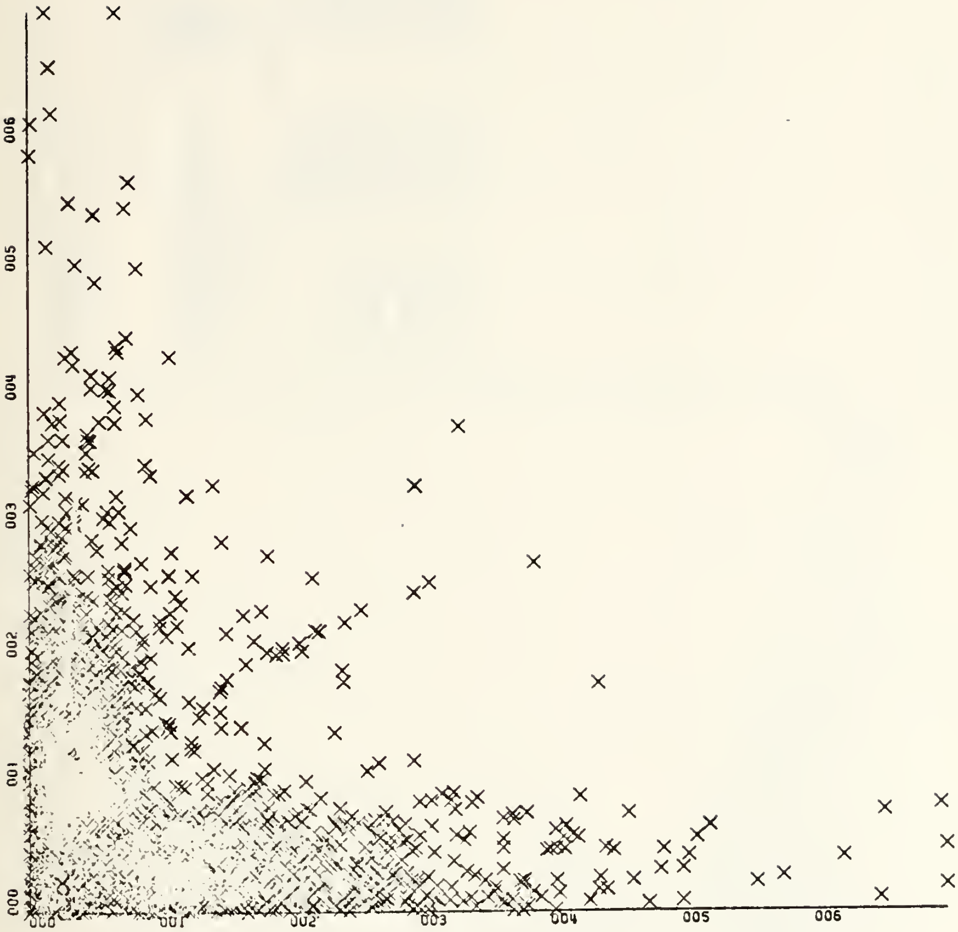


Figure V-c3. Scatter plot for sample of size 2000 from mixture-truncation bivariate exponential with $\rho = -0.3$. Here x_o has triangular distribution between the lower and the upper bounds.

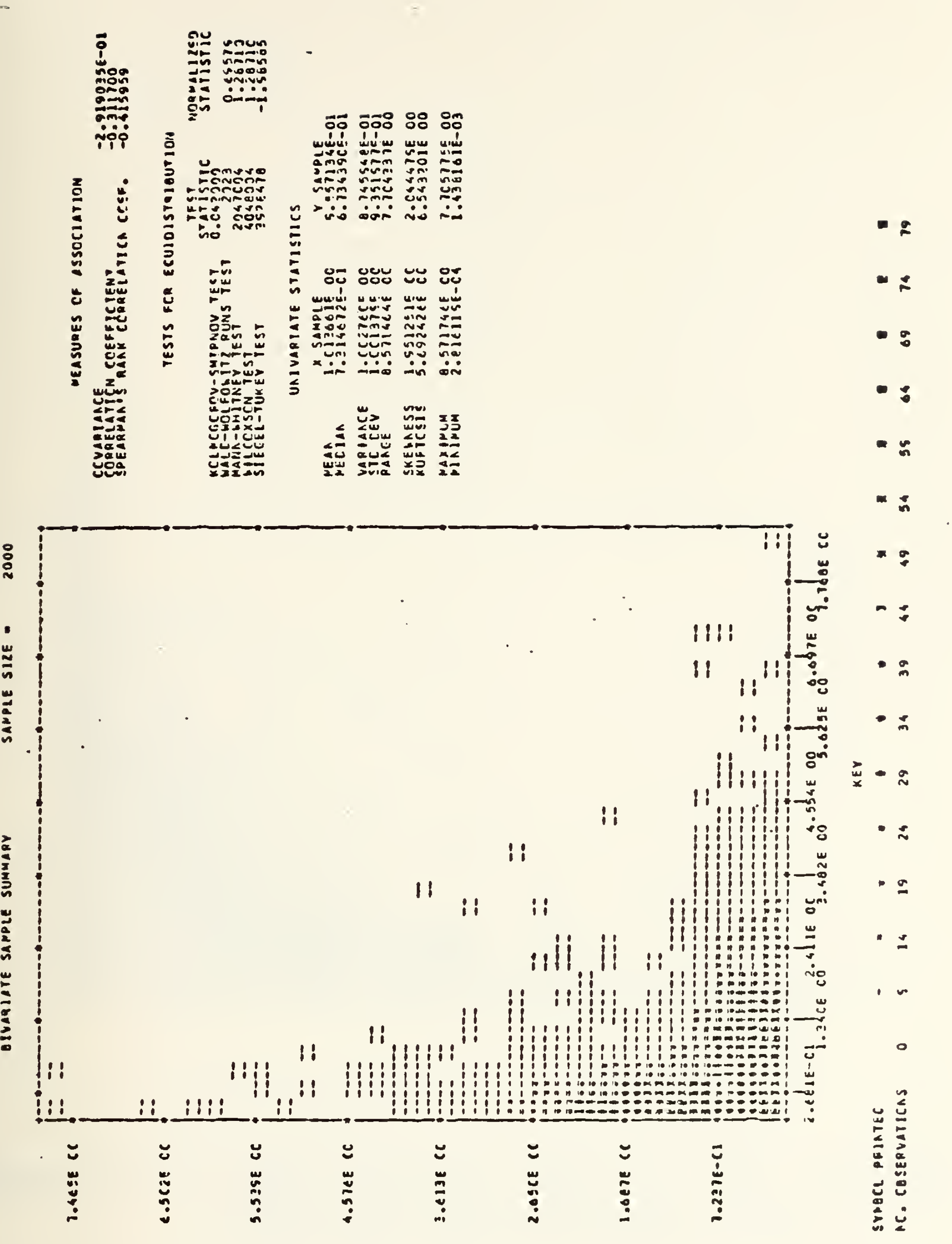


Figure V-c4. Bivariate historgram for sample of size 2000 from mixture-truncation bivariate with $\rho = -0.3$. Here x_o has triangular distribution between the lower and the upper bounds.

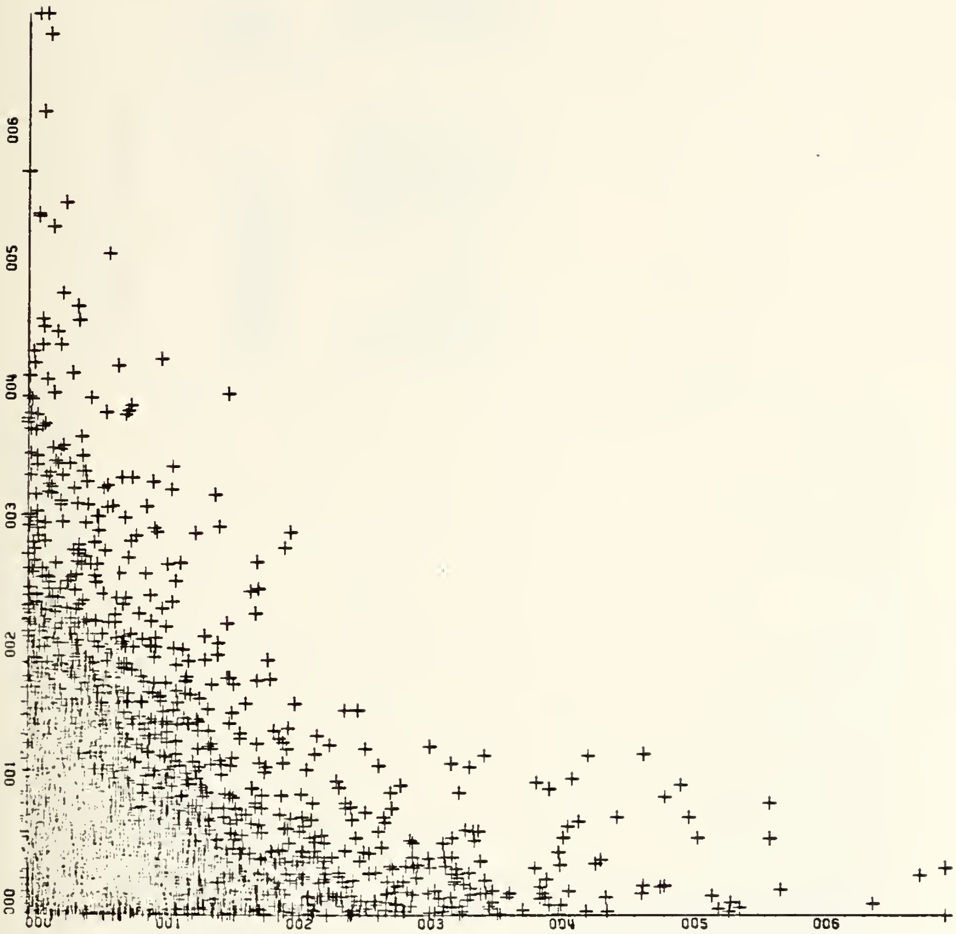


Figure V-dl. Scatter plot for sample of size 2000 from Gaver's bivariate exponential with $\rho = -0.3$.

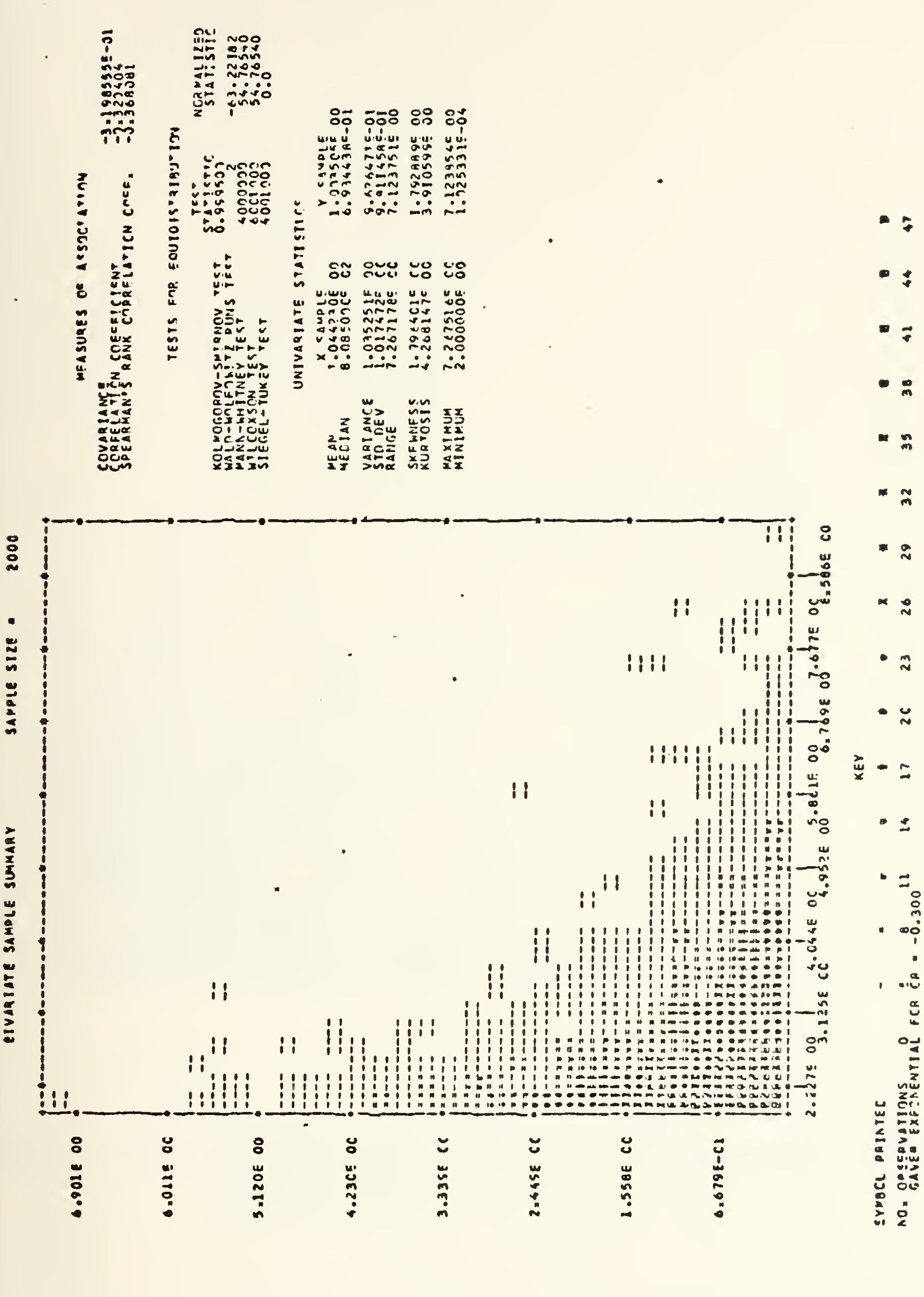


Figure V-d2. Bivariate histogram for sample of size 2000 from Gaver's bivariate exponential with ρ = -0.3.

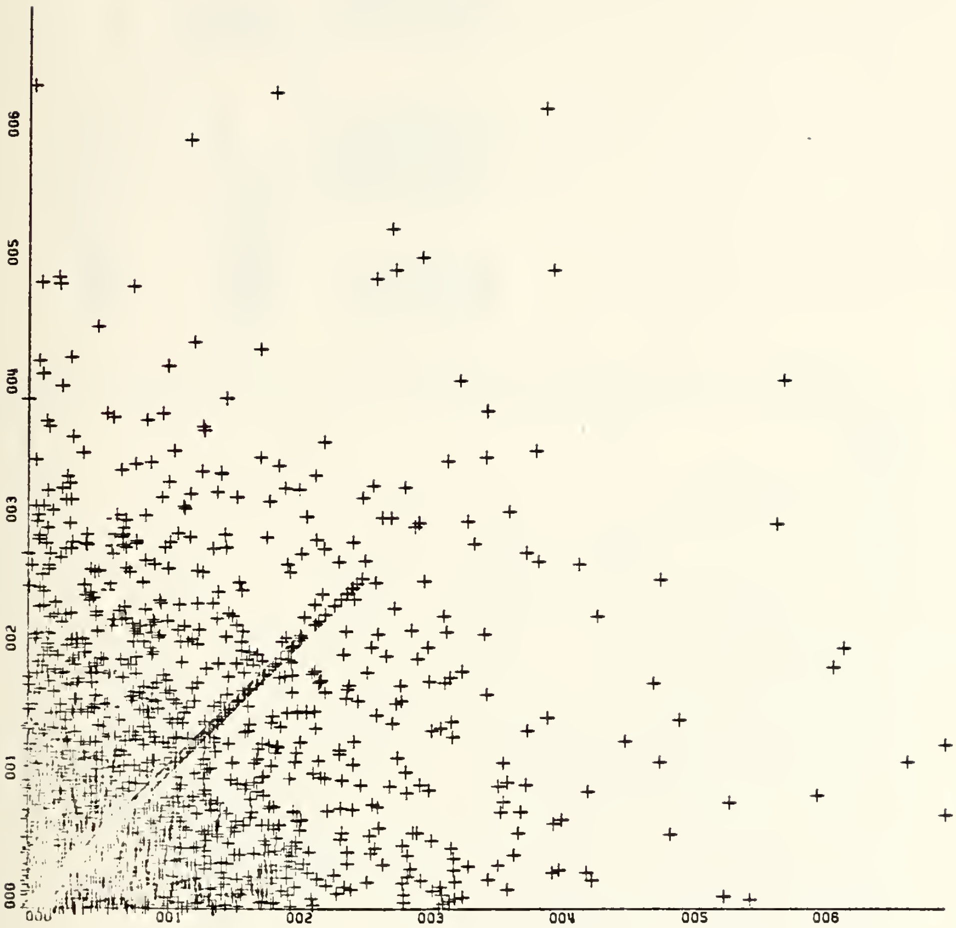


Figure V-el. Scatter plot for sample of size 2000 from Marshall and Olkin's bivariate exponential with $\rho = 0.3$.

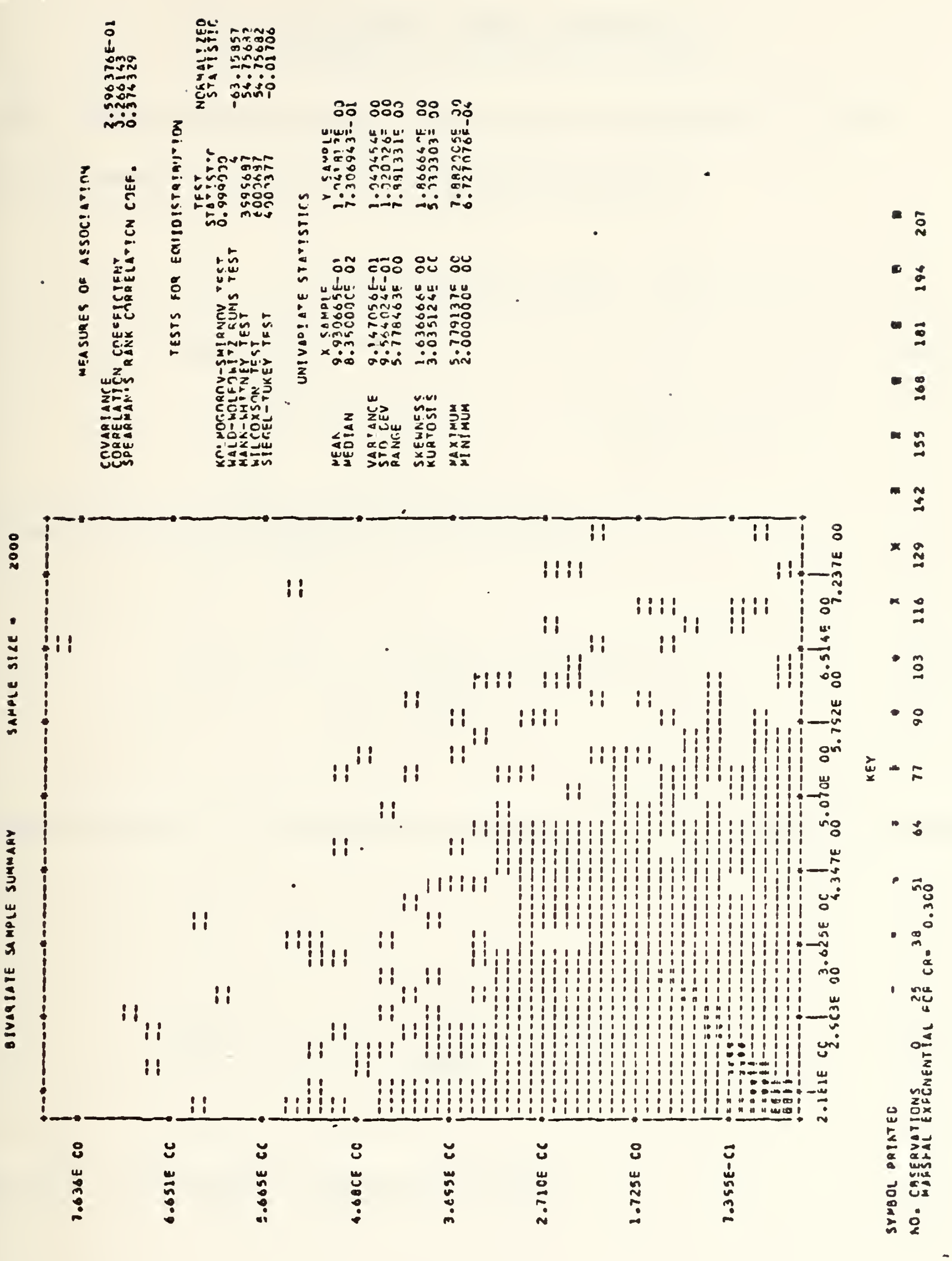


Figure V-e2. Bivariate histogram for sample of size 2000 from Marshall and Olkin's bivariate exponential with ρ = 0.3.

VI. BIVARIATE GAMMA GENERATOR

The gamma distribution with shape parameter r and scale parameter λ has the density function

$$f(x) = \begin{cases} \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \tag{VI-0-1}$$

and cumulative distribution function

$$F(y) = \int_0^y \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x} dx . \tag{VI-0-2a}$$

In particular, if the shape parameter r is a positive integer then

$$F(y) = 1 - \sum_{j=0}^{r-1} \frac{e^{-\lambda y}(\lambda y)^j}{j!} , \tag{VI-0-2b}$$

where $\Gamma(r)$ is Euler's gamma function

$$\Gamma(r) = \int_0^\infty x^{r-1} e^{-x} dx .$$

If the random variable X has Gamma (r,λ) distribution then

$$E[x) = \frac{r}{\lambda}$$

$$VAR[x] = \frac{r}{\lambda^2} \quad ;$$

when r = 1, X has the exponential distribution while X, suitably scaled, has an asymptotically normal distribution as $r \to \infty$. We note that if X has a Gamma(r,1) distribution then $\frac{X}{\lambda}$ has a Gamma(r,λ) distribution. So we may set = 1 as far as the generating algorithm is concerned. The output from the generator may then be appropriately scaled. Two algorithms for generating bivariate gamma random vectors are given in Section II. Since they have complicated computations to define the parameter values and use inverse transformation functions, the mixture-truncation method is probably preferable to those methods. To generate bivariate random vectors with identical Gamma (r,1) marginal distributions and having given correlation ρ by mixture-truncation method, we only need univariate gamme (r,1) random variate generator. Under the assumption that the univariate gamma generator is available we developed this procedure. In fact we used Lewis and Robinson's gamma generator as a univariate gamma generator.

A. DETERMINATION OF PARAMETERS IN THE GAMMA MIXTURE-TRUNCATION METHOD

If the marginal distribution is gamma with positive integer shape parameter r and scale parameter $\lambda = 1$, then

we know that $F(x)$ becomes cumulative distribution function of gamma (r,1). Then the X_1 is gamma (r,1) truncated to the left of x_o and X_2 is also gamma (r,1) truncated to the right of x_o. Thus we have

$$E[X_1] = \mu_1 = \int_0^{x_o} x \, d \frac{F(x)}{F(x_o)} dx$$

$$= \frac{1}{F(x_o)\Gamma(r)} \int_0^{x_o} x^r e^{-x} dx$$

$$= \frac{1}{\pi_1 \Gamma(r)} \{r! - e^{-x_o} [\sum_{i=0}^{r} \frac{r!}{(r-i)!} x_o^{r-i}]\}$$

$$VAR[X_1] = \sigma_1^2 = \int_0^{x_o} x^2 \, d \frac{F(x)}{F(x_o)} - \mu_1^2$$

$$= \frac{1}{\pi_1 \Gamma(r)} \int_0^{x_o} x^{r+1} e^{-x} dx - \mu_1^2$$

$$= \frac{1}{\pi_1 \Gamma(r)} \{(r+1)! - e^{-x_o} [\sum_{i=0}^{r+1} \frac{(r+1)!}{(r+1-i)!} x_o^{r+1-i}]\}$$

$$- \mu_1^2$$

$$E[X_2] = \mu_2 = \int_{x_o}^{\infty} x \, d \frac{F(x) - F(x_o)}{1 - F(x_o)}$$

$$= \frac{1}{\pi_2 \Gamma(r)} \int_{x_o}^{\infty} x^r e^{-x} \, dx$$

$$= \frac{1}{\pi_2 \Gamma(r)} [e^{-x_o} \sum_{i=0}^{r} \frac{r!}{(r-i)!} x_o^{r-i}]$$

$$VAR[X_2] = \sigma_2^{\,2} = \int_{x_o}^{\infty} x^2 \, d \frac{F(x) - F(x_o)}{1 - F(x_o)}$$

$$= \frac{1}{\pi_2 \Gamma(r)} \int_{x_o}^{\infty} x^{r+1} e^{-x} \, dx$$

$$= \frac{1}{\pi_2 \Gamma(r)} [e^{-x_o} \sum_{i=0}^{r+1} \frac{(r+1)!}{(r+1-i)!} x_o^{r+1-i}] - \mu_2^{\,2}$$

Let

$$A = e^{-x_o} \sum_{i=0}^{r} \frac{r}{(r-i)!} x_o^{r-i}$$

$$B = e^{-x_o} \sum_{i=0}^{r+1} \frac{(r+1)!}{(r+1-i)!} x_o^{r+1-i}$$

and substitute, then we have

$$E[X_1] = \mu_1 = \frac{r! - A}{\pi_1 \Gamma(r)} ,$$

$$VAR[X_1] = \sigma_1^2 = \frac{(r+1)! - B}{\pi_1 \Gamma(r)} - \frac{(r! - A)^2}{\pi_1^2 \Gamma(r)^2} ,$$

$$E[X_2] = \mu_2 = \frac{A}{\pi_2 \Gamma(r)} ,$$

$$VAR[X_2] = \sigma_2^2 = \frac{B}{\pi_2 \Gamma(r)} - \frac{A^2}{\pi_2^2 \Gamma(r)^2} .$$

If we use these formulae in Theorem 2 in Section III then we have

$$M = \frac{(\pi_2 r! - A)^2}{\pi_1 \pi_2 (\Gamma(r))^2 r}$$

and

$$\beta = \frac{\alpha_1 - \pi_1}{\pi_2} ,$$

Thus

$$\rho = \frac{(\alpha_1 - \pi_1)(\pi_2 r! - A)^2}{r\ \pi_1\ \pi_2^2 (\Gamma(r))^2} . \qquad \text{(VI-A-1)}$$

For given ρ, x_o and r,

$$\alpha_1 = \frac{\rho\ r!\ (r-1)!\ \pi_1\ \pi_2^2}{(\pi_2 r! - A)^2} + \pi_1 \qquad \text{(VI-A-2a)}$$

and

$$\alpha_2 = \frac{\pi_2 - \pi_1(1 - \alpha_1)}{\pi_2} . \qquad \text{(VI-A-2b)}$$

The value of (VI-A-2a) and (VI-A-2b) are determined with given ρ, x_o and r. Thus for given marginal distribution and correlation coefficient ρ we can find x_o ranges as before. For simplicity, we will consider them with marginal distribution gamma (2,1). From the formula of (IV-0-2b),

$$\pi_1 = F(x_o) = 1 - e^{-x_o} - x_o e^{-x_o}$$

$$\pi_2 = 1 - \pi_1 = e^{-x_o} + x_o e^{-x_o}$$

and

$$A = e^{-x_o}(x_o^2 + 2x_o + 2) .$$

Then (VI-A-2a) and (VI-A-2b) become

$$\alpha_1 = \frac{2 \rho \pi_1 \pi_2^2}{x_o^4 e^{-2x_o}} + \pi_1 \qquad \text{(VI-A-3a)}$$

$$\alpha_2 = \pi_2 + \frac{2 \rho \pi_1^2 \pi_2}{x_o^4 e^{-2x_o}} . \qquad \text{(VI-A-3b)}$$

These α_1 and α_2 are probabilities, so they have to be greater than or equal to zero and less than or equal to one;

$$0 \leq \alpha_1 = \frac{2\rho\pi_1\pi_2^2}{x_o^4 e^{-2x_o}} + \pi_1 \leq 1$$

$$0 \leq \alpha_2 = \pi_2 + \frac{2\rho\pi_1^2\pi_2}{x_o^4 e^{-2x_o}} \leq 1$$

From these two inequality equations we can find the x_o ranges for given correlation coefficient ρ. If $\rho \geq 0$, then α_1 and α_2 are always positive. Thus we only need to find the x_o ranges which makes (VI-A-3a) and (VI-A-3b) less than one. From equation (VI-A-3a), the α_1 case, and from equation (VI-A-3b), the α_2 case, we have exactly the same inequality equation;

$$2\rho e^{x_o}(1 + x_o) \leq x_o^4 + 2\rho\,(1 + x_o)^2 \qquad \text{(VI-A-4)}$$

If $\rho \leq 0$, then α_1 and α_2 are always less than one. Thus we only need to find the x_o ranges which makes (VI-A-3a) and (VI-A-3b) greater than zero. From the α_1 equation, we have

$$\sqrt{2\rho^*}(1 + x_o) \leq x_o^2 \qquad \text{(VI-A-5a)}$$

where

$$\rho^* = |\rho|$$

and from the α_2 equation, we have

$$\sqrt{2\rho^*}\, e^{x_o} \leq x_o^{\,2} + \sqrt{2\rho^*}(1 + x_o) \qquad \text{(VI-A-5b)}$$

where

$$\rho^* = |\rho| .$$

Note that if we let y_1 for the left side equation of inequality sign and y_2 for the right side equation of inequality equations (VI-A-4), (VI-A-5a) and (VI-A-5b), then we know that all y_1 and y_2 are monotone increasing functions. Thus as in the exponential case we can find the range of x_o by the Newton Raphson method. For the positive cirrelation case, equation (VI-A-4) will give the x_ℓ and x_u and for the negative correlation case, equation (VI-A-5a) gives x_ℓ and equation (VI-A-5b) gives x_u. By using subroutine GBOUND which is listed in the appendix, we can find x_ℓ and x_u. The tables (VI-a) and (VI-b) show the results from subroutine GBOUND. Unfortunately we have limitations on the possible range of correlations; we can generate bivariate random vectors with correlations in approximately the range $-.5 \leq \rho \leq .6$ for the Gamma (2,1) marginal distribution. In the same way as the above, the values x_ρ and x_u can be computed with increasing complexity for integer values of r greater than 2. In principle it is also possible, using numerical integration, to compute allowable values for non-integer values of r. The computation is not as complicated as those required for Schmeiser's bivariate gamma. Moreover the mixture-truncation method does not require that the user be able to compute the inverse gamma distribution function.

Table VI-a: x_o ranges with Gamma (2,1) marginal distribution and given ρ

ρ	x_L	x_u	ρ	x_L	x_u
0.1	0.41	7.38	-0.1	0.93	3.41
0.2	0.65	6.23	-0.2	1.18	2.75
0.3	0.89	5.42	-0.3	1.35	2.34
0.4	1.15	4.74	-0.4	1.50	2.03
0.5	1.47	4.09	-0.5	1.62	1.79
0.6	1.95	3.32			

Table VI-b: x_o ranges with Gamma (3,1) marginal distribution and given ρ

ρ	x_L	x_u	ρ	x_L	x_u
0.1	0.82	8.99	-0.1	1.64	4.73
0.2	1.17	7.72	-0.2	1.37	3.98
0.3	1.49	6.82	-0.3	2.21	3.5
0.4	1.83	6.05	-0.4	2.4	3.15
0.5	2.24	5.31	-0.5	2.56	2.89
0.6	2.84	4.42			

B. GENERATING PROCEDURE

The generating procedure of bivariate gamma random vectors is almost the same as the uniform and exponential cases. We also developed here three methods which are the FXO method, the UXO method and the TXO method. As in the uniform and exponential cases, these three methods are the same except in the choice of x_o from the x_o range $[x_\ell, x_u]$. The FXO method chooses x_o as a fixed point from the x_o range and uses it during the entire routine while the UXO and TXO methods choose another x_o in every routine. From experience, the midpoint of the x_o range gives the best result of the FXO method. In the UXO method and the TXO method, we assume that x_o has the uniform distribution and triangular distribution, respectively, over $[x_\ell, x_u]$. This assumption removes some discontinuity which occurs at the truncation point in the FXO method.

<u>Gamma Mixture-Truncation Method</u>
<u>(for integer shape parameters)</u>

1. (Initialization)

   i) For given allowable correlation coefficient ρ, find x_ℓ and x_u

2. Define truncation point x_o

   * FXO method

   i) $x_o = \frac{1}{2}(x_\ell + x_u)$

* UXO method

  i) Generate a uniform [0,1] random variable U_1

  ii) $x_o = x_\ell + (x_u - x_\ell) * U_1$

* TXO method

  i) Generate two uniform [0,1] random variables, V_1, V_2

  ii) $x_o = x_\ell + x_1 + x_2$

  where

$$x_m = (x_\ell + x_u)/2.0 ,$$

$$x_1 = (x_m - x_\ell) * V_1 ,$$

$$x_2 = (x_u - x_m) * V_2 .$$

3. Compute parameter values

$$\pi_1 = F(x_o) = 1 - \sum_{j=0}^{r-1} \frac{e^{-x_o} x_o^{\,j}}{j!} ,$$

$$\pi_2 = 1 - \pi_1 ,$$

$$\alpha_1 = \frac{\rho\, r!\,(r-1)!\,\pi_1 \pi_2^2}{(\pi_2 r! - A)^2} + \pi_1 ,$$

$$\alpha_2 = 1 - \frac{\pi_1}{\pi_2}(1 - \alpha_1) ,$$

where

$$A = e^{-x_o} \sum_{i=0}^{r} \frac{r}{(r-i)!} x_o^{r-i}$$

4. Choose type for Y

   i) Generate a uniform [0,1] random variable U

   ii) If $U \leq \pi_1$, go to 9

5. Y is an X_2

   i) Generate a gamma(r,1) random variable G_1

   ii) If $G_1 > x_o$, set $Y \leftarrow G_1$ and go to 6

   iii) Otherwise return to 5.i)

6. Choose type for Z

   i) Set $U \leftarrow ((U - \pi_1)/(1 - \pi_1))$

   ii) If $U \leq 1 - \alpha_2$, go to 8

7. Z is an X_2

   i) Generate a gamma (r,1) random variable G_2

   ii) If $G_2 > x_o$, set $Z \leftarrow G_2$ and go to 11

   iii) Otherwise return to 7.i)

8. Z is an X_1

   i) Generate a gamma (r,1) random variable G_2

   ii) If $G_2 \leq x_o$, set $Z \leftarrow G_2$ and go to 11

   iii) Otherwise return to 8.i)

9. Y is an X_1

  i) Generate a gamma (r,1) random variable G_1

  ii) If $G_1 \leq x_o$, set $Y \leftarrow G_1$ and go to 10

  iii) Otherwise return to 9.i)

10. Choose type for Z

  i) Set $U \leftarrow U/\pi_1$

  ii) If $U \leq \alpha_1$, go to 8

  iii) Otherwise go to 7

11. Deliver (Y,Z) and go to 4 for the FXO method, or go to 2 for the UXO and TXO methods until a sufficient number of random vectors are obtained.

For the non-integer shape parameter case, step 1 and step 3 need more complicated computation procedures but here we are only concerned for integer shape parameter cases. In step 1 we used the subroutine GBOUND which is listed in the appendix. The scatter plots and bivariate histograms of the resulting random vectors from this algorithm are shown in the next section.

C. SIMULATION RESULTS

The scatter plot and bivariate histograms for random vectors of the mixture-truncation bivariate gamma variables with marginal gamma (2,1) distribution and given correlation (0.3 and -0.3) are given here. As in the uniform and

exponential cases, the UXO, TXO and FXO methods are used and are shown in Figures (VI-a), (VI-b) and (VI-c), respectively.

The FXO method has some discontinuity at the truncation point, but the UXO and TXO methods appear to give a relatively smooth continuous distribution. The computed correlation in subroutine BIVHST (Bivariate histogram) is a little different from the given correlation. But we can assume this difference is a result of sampling error.

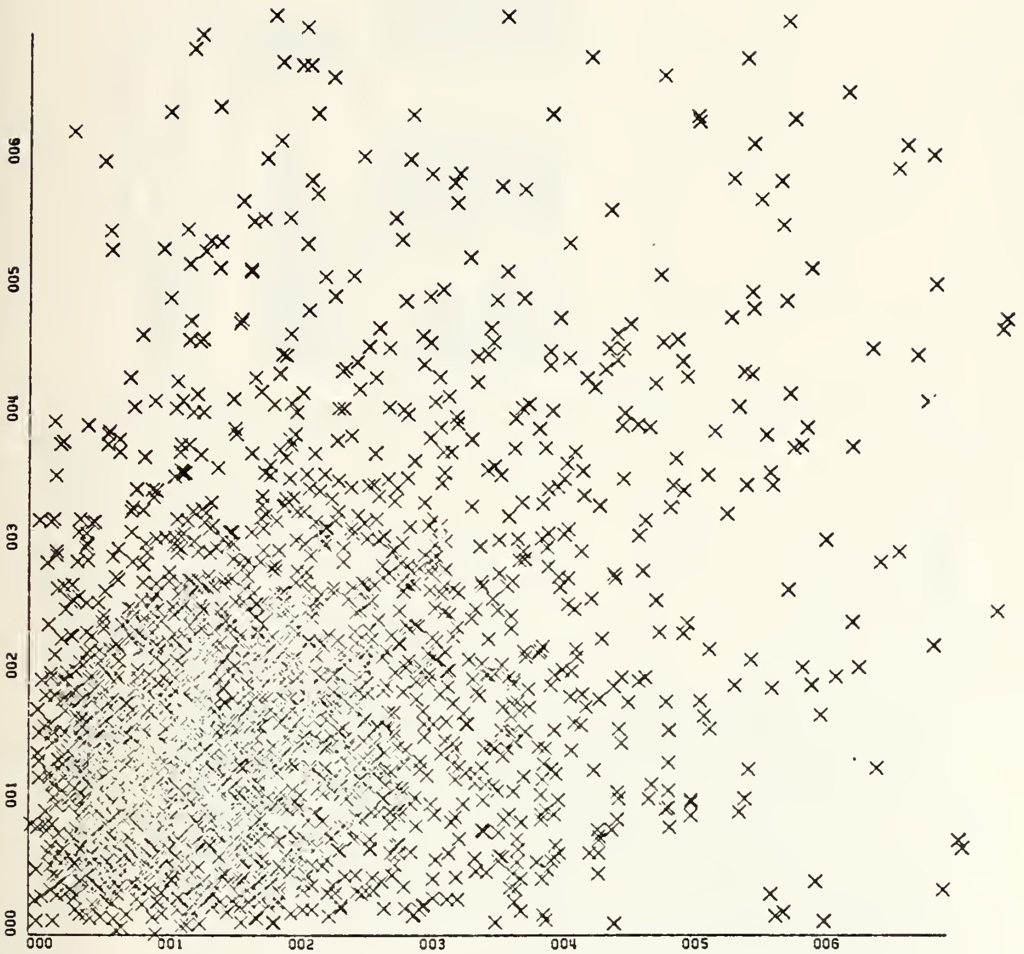


Figure VI-a1. Scatter plot for sample of size 2000 from mixture-truncation bivariate Gamma distribution with $\rho = 0.3$. Here x_o has uniform distribution over (x_ℓ, x_u).

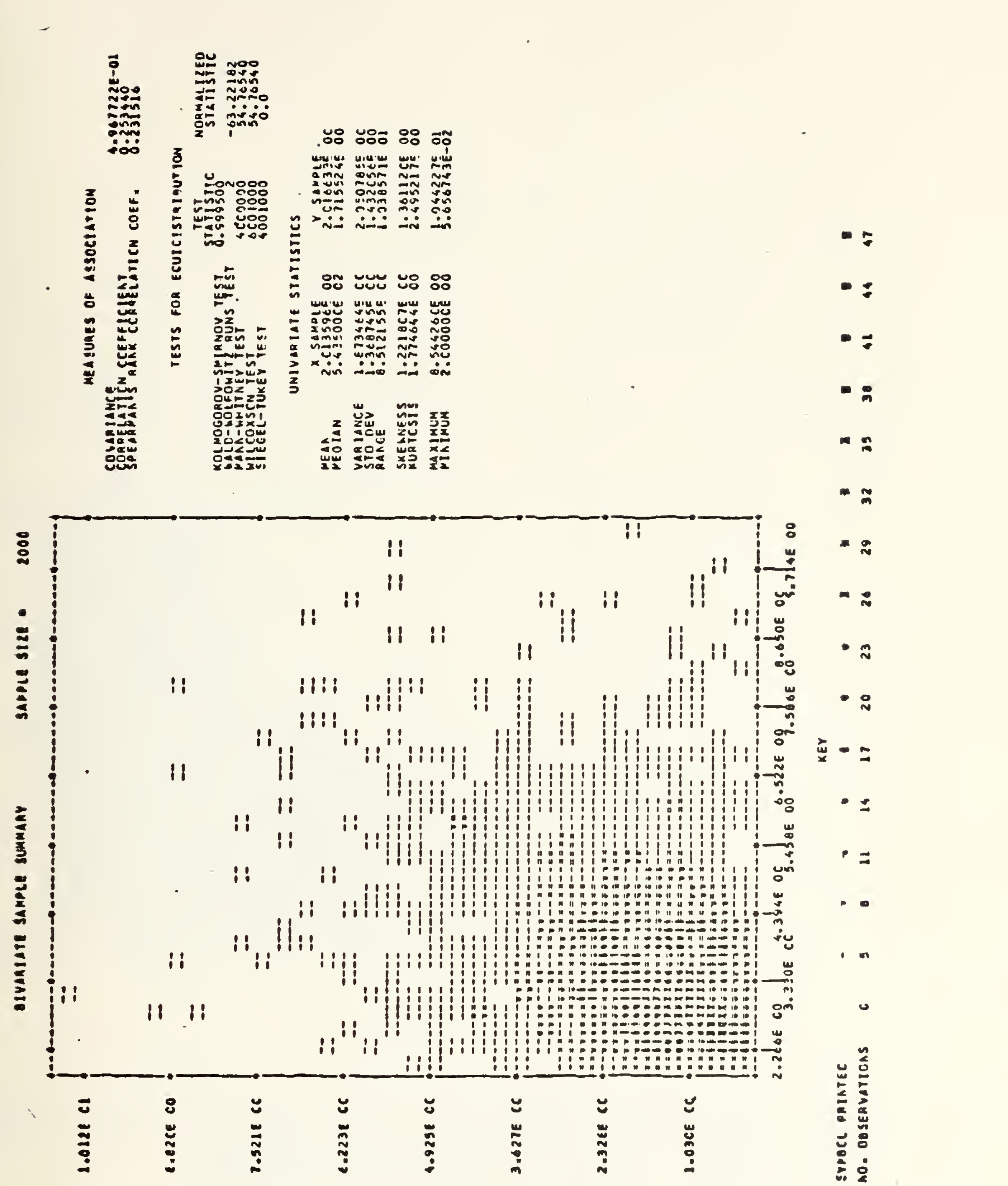


Figure VI-a2. Bivariate histogram for sample of size 2000 from mixture-truncation bivariate Gamma distribution with $\rho = 0.3$. Here x_o has uniform distribution over (x_ℓ, x_u).

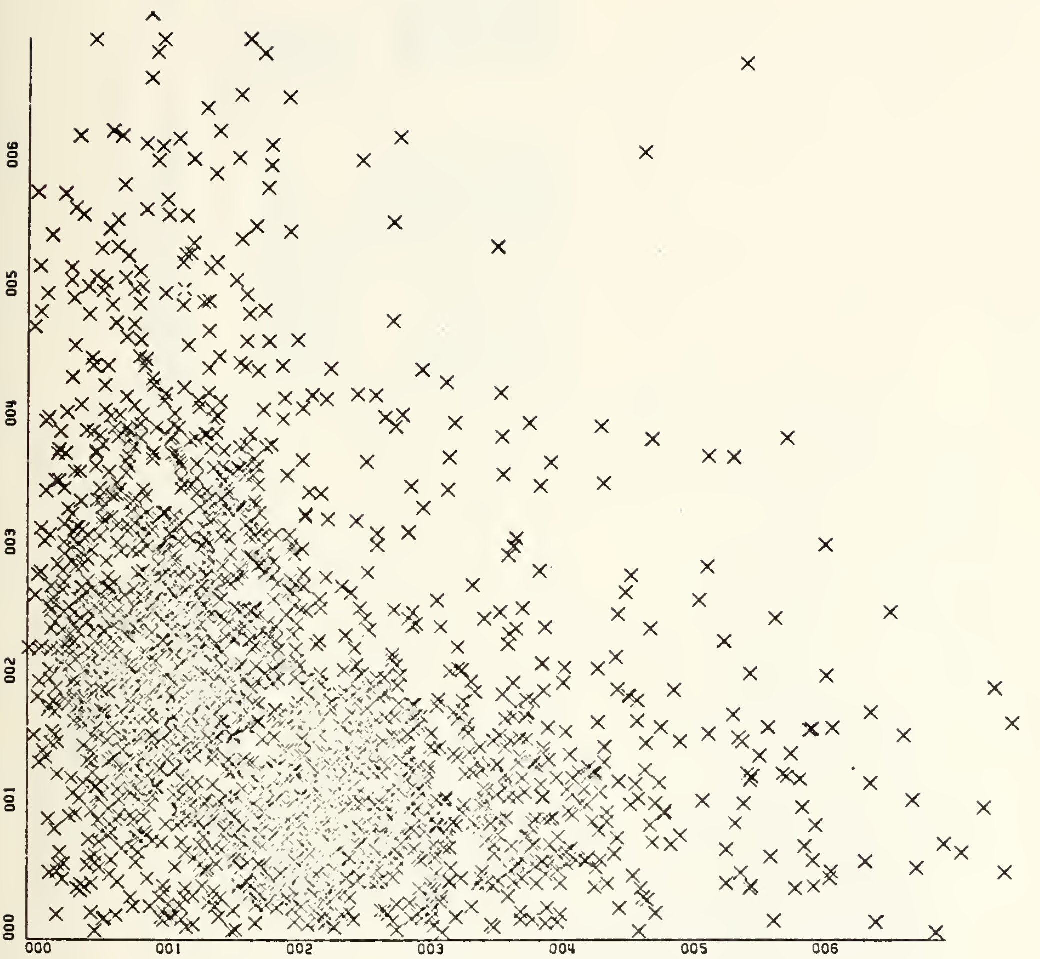


Figure VI-a3. Scatter plot for sample of size 2000 from mixture-truncation bivariate Gamma distribution with $\rho = -0.3$. Here x_o has uniform distribution over (x_ℓ, x_u).

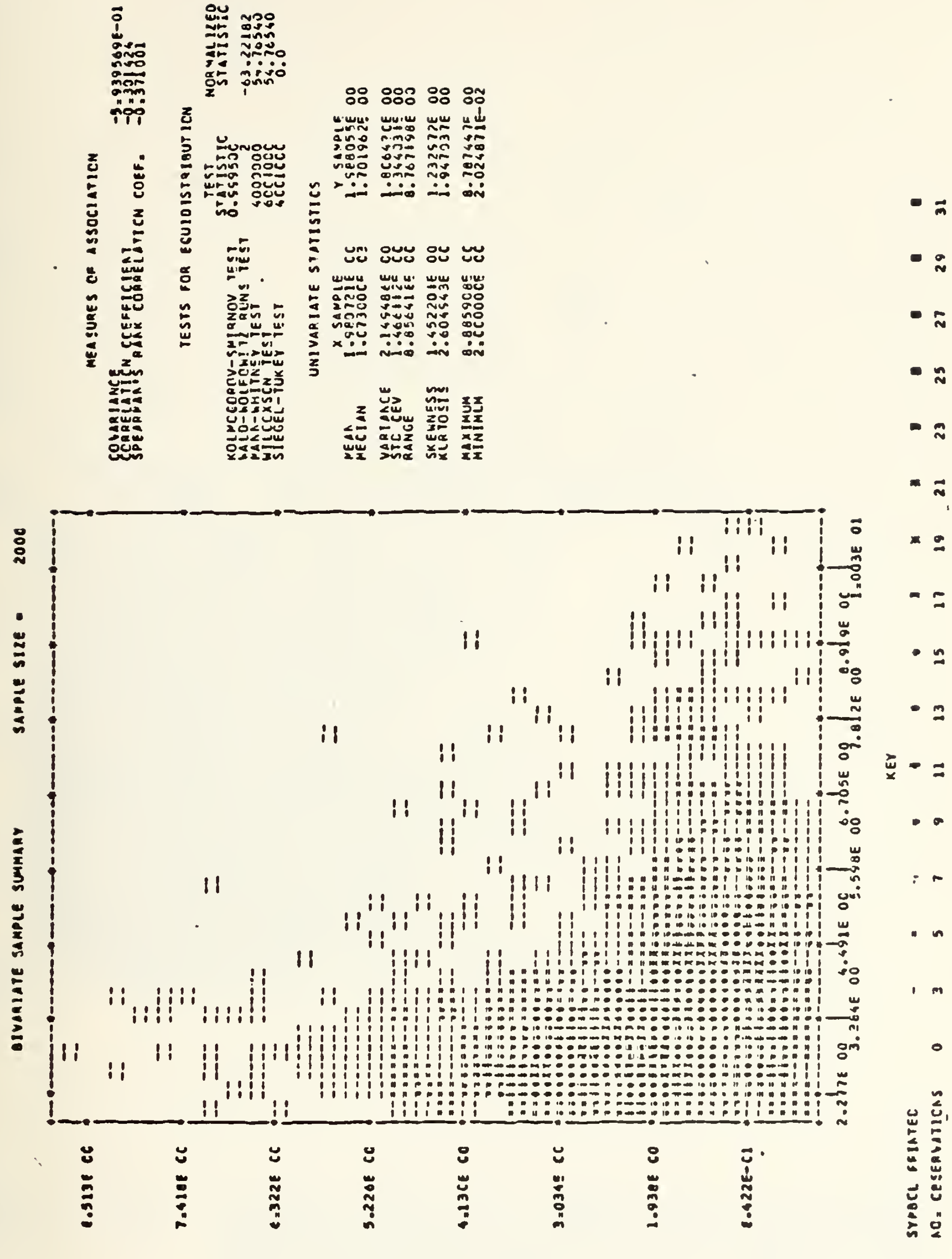


Figure VI-a4. Bivariate histogram for sample of size 2000 from mixture-truncation bivariate Gamma distribution with $\rho = -0.3$. Here x_o has uniform distribution over (x_ℓ, x_u).

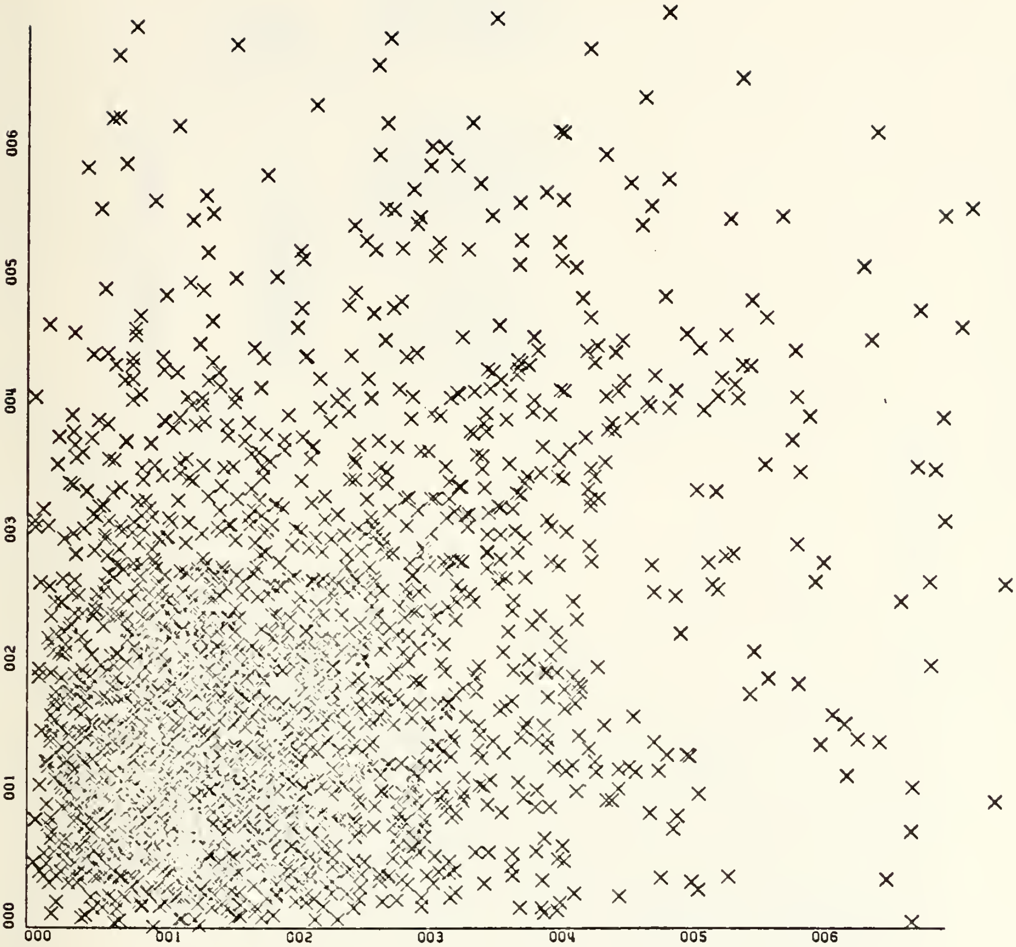


Figure VI-b1. Scatter plot for sample of size 2000 from mixture truncation bivariate Gamma distribution with $\rho = 0.3$. Here x_o has triangular distribution over (x_ℓ, x_u^o).

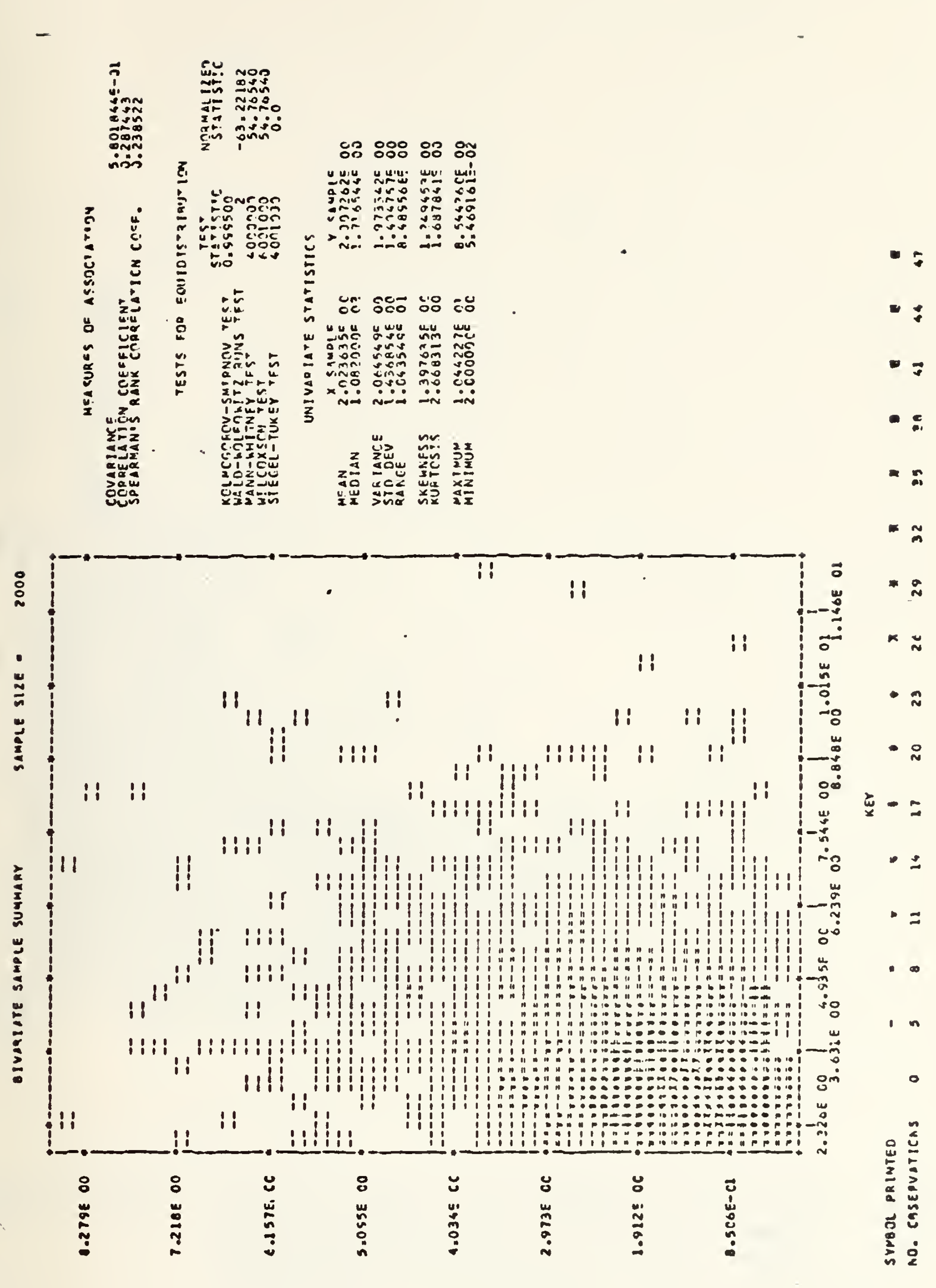


Figure VI-b2. Bivariate histogram for sample of size 2000 from mixture-truncation bivariate Gamma distribution with $\rho = 0.3$. Here x_o has triangular distribution over (x_ℓ, x_u).

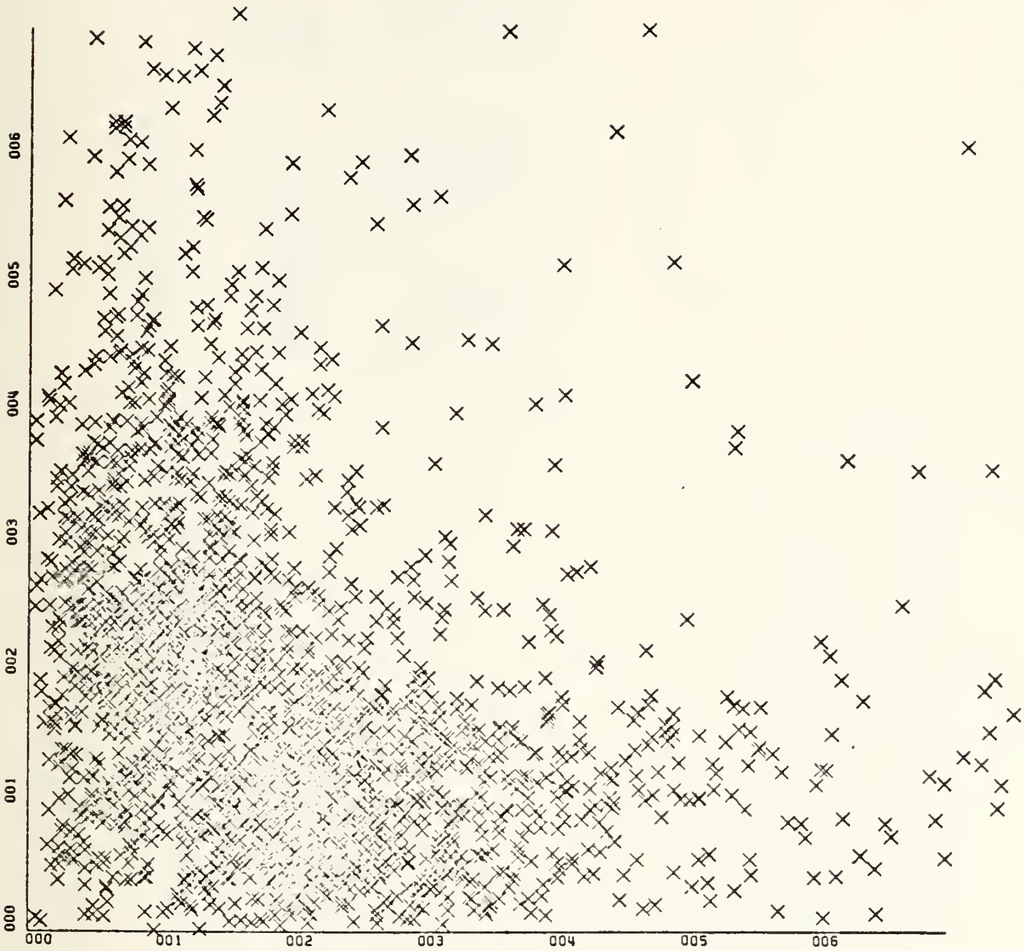


Figure VI-b3. Scatter plot for sample of size 2000 from mixture-truncation bivariate Gamma distribution with $\rho = -0.3$. Here x_{ϱ} has triangular distribution over (x_{ℓ}, x_u).

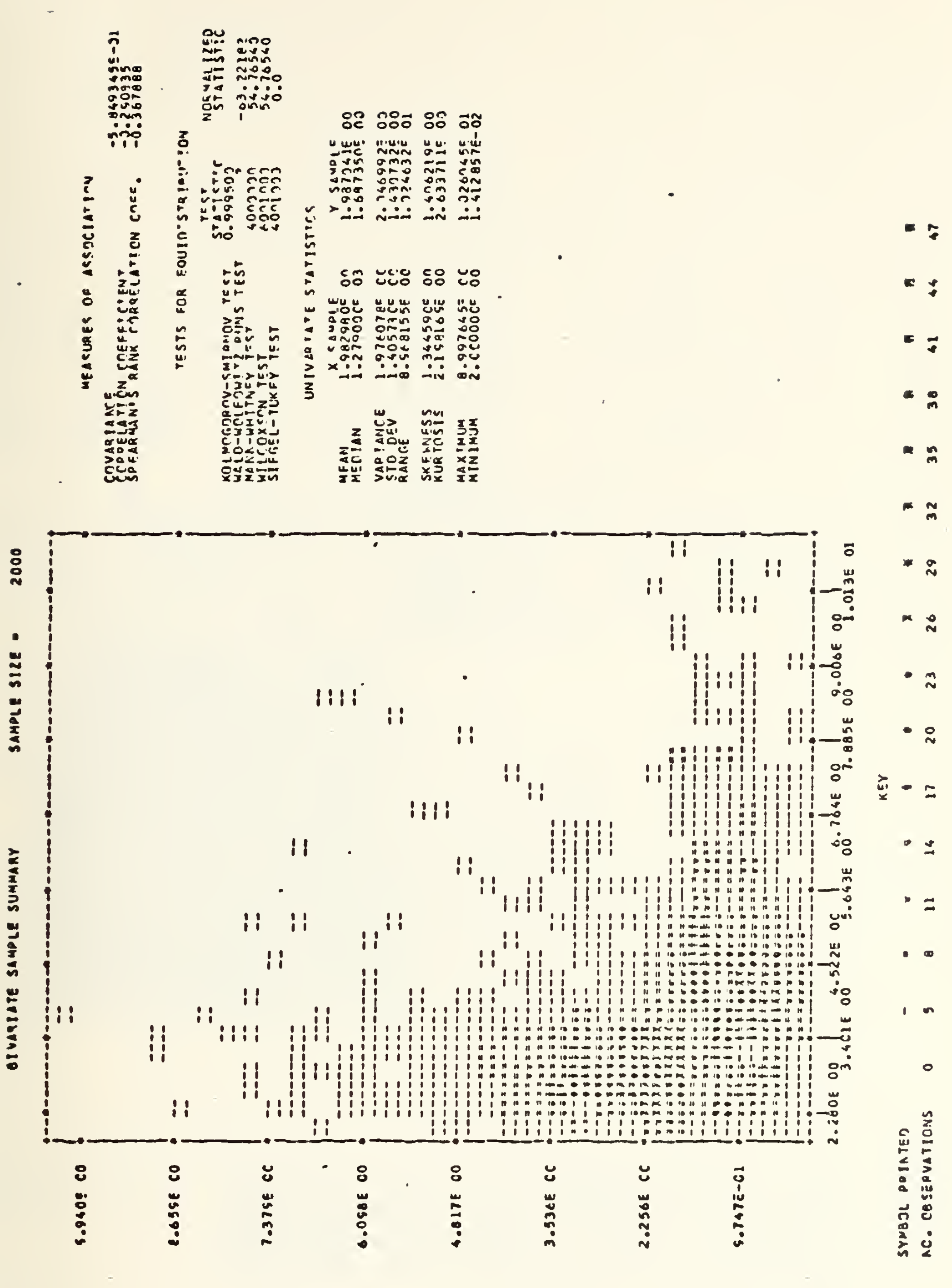


Figure VI-b4. Bivariate histogram for sample of size 2000 from mixture-truncation bivariate Gamma distribution with $\rho = -0.3$. Here x_o has triangular distribution over (x_ℓ, x_u).

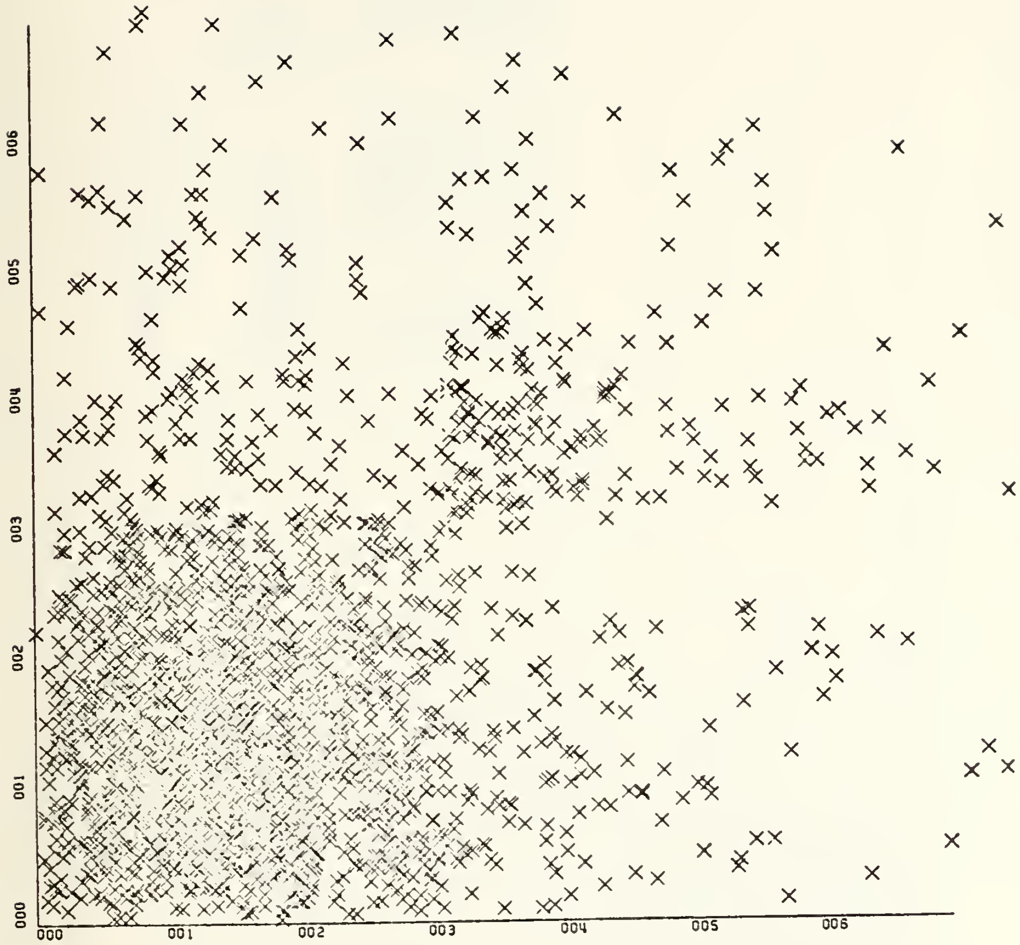


Figure VI-c1. Scatter plot for sample of size 2000 from mixture-truncation bivariate Gamma distribution with $\rho = 0.3$. Here x_o is fixed at the midpoint of x_ℓ and x_u.

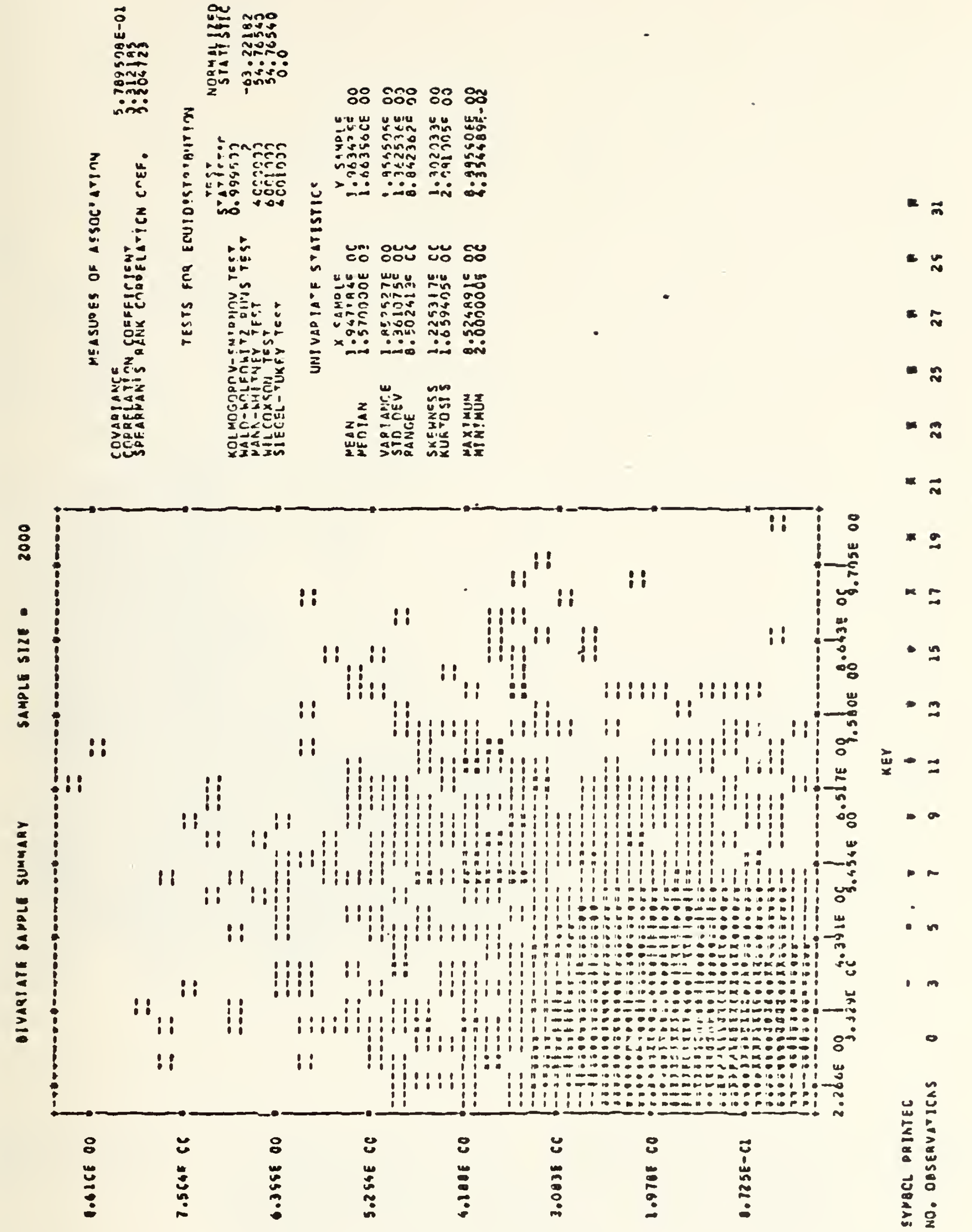


Figure VI-c2. Bivariate histogram for sample of size 2000 from mixture-truncation bivariate Gamma distribution with $\rho = 0.3$. Here x_o is fixed at the midpoint of x_ℓ and x_u.

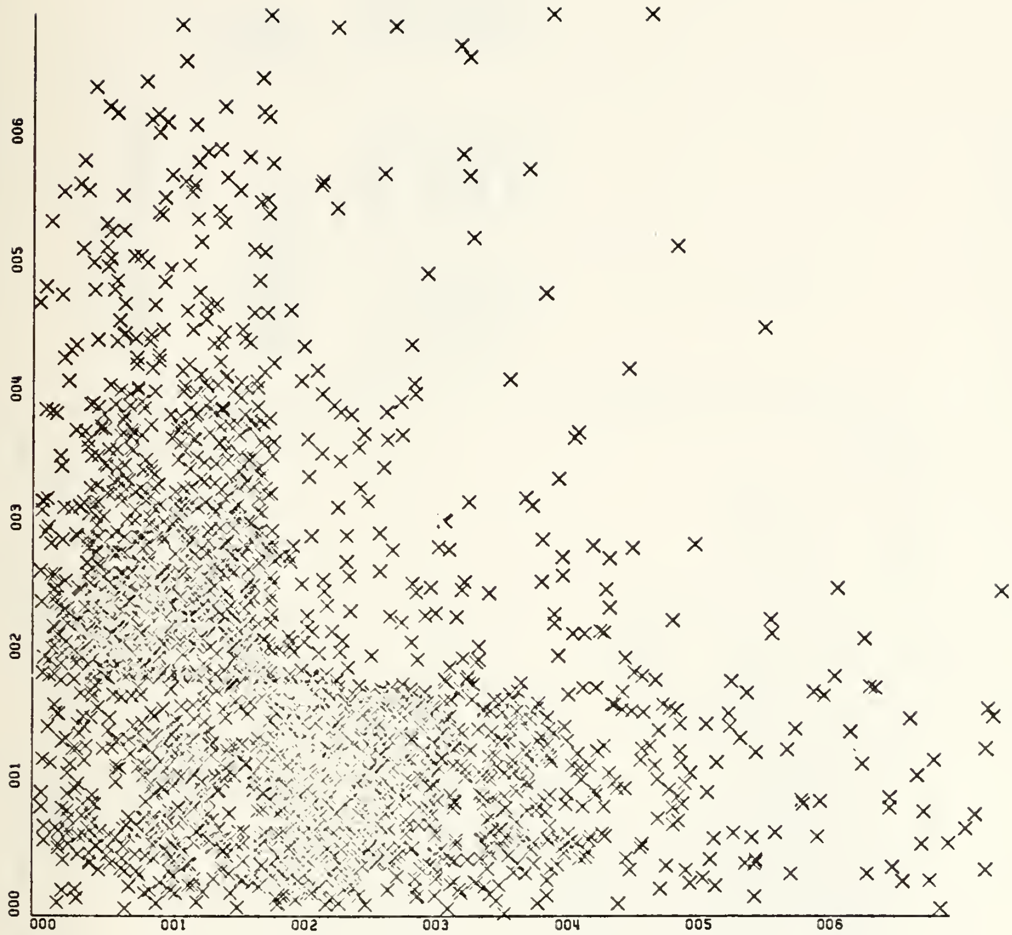


Figure VI-c3. Scatter plot for sample of size 2000 from mixture truncation bivariate Gamma distribution with $\rho = -0.3$. Here x_o is fixed at the midpoint of x_ℓ and x_u.

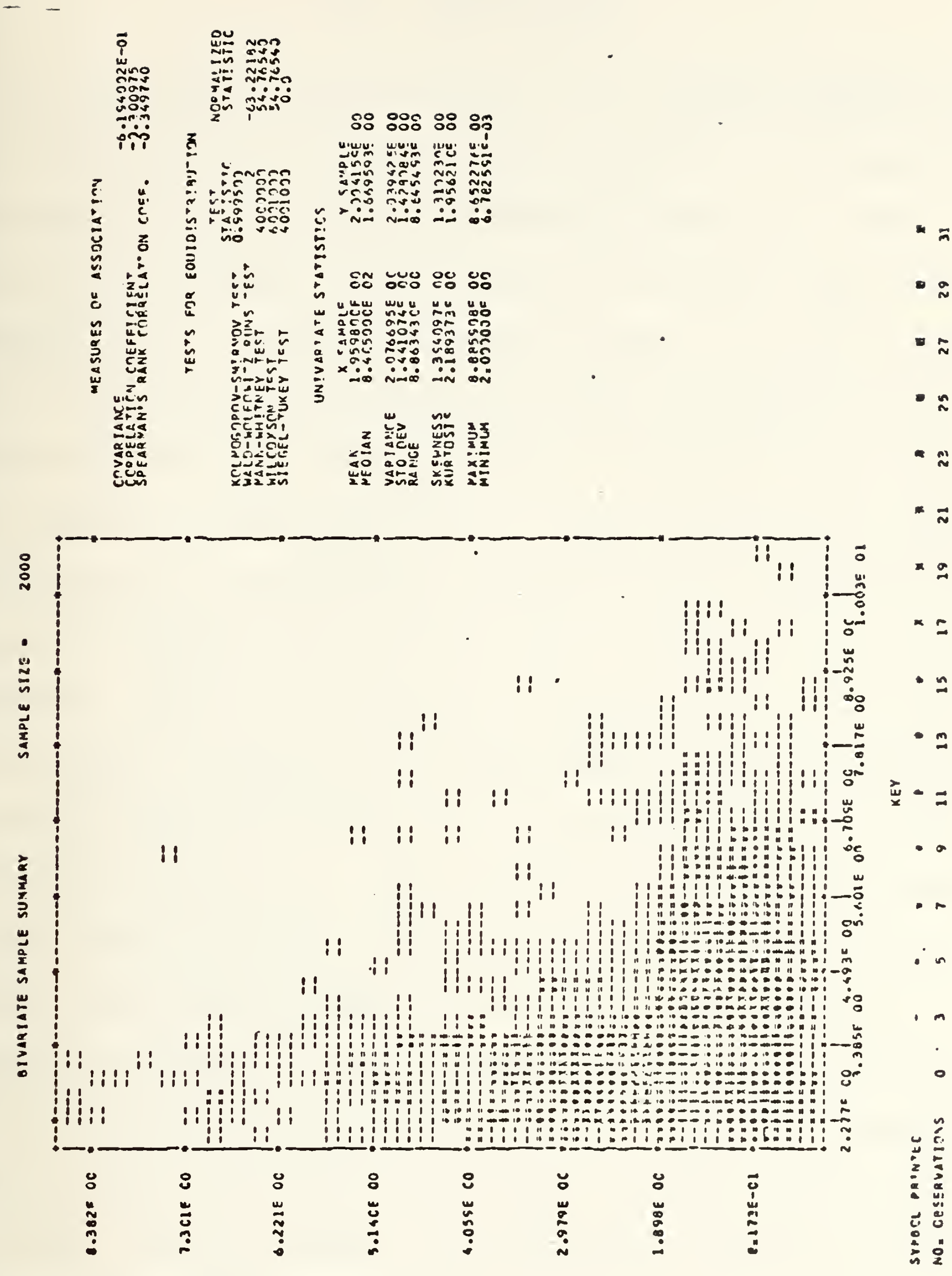


Figure VI-c4. Bivariate histogram for sample of size 2000 from mixture-truncation bivariate Gamma distribution with $\rho = -0.3$. Here x_o is fixed at the midpoint of x_ℓ and x_u.

VII. CONCLUSION

The mixture-truncation method is a general method which can generate bivariate random vectors having any theoretical marginal distribution and allowable correlation. The generating procedure is very simple and doesn't need much computation for defining parameter values. In this respect, the mixture-truncation method is a very attractive method for generating bivariate random vectors. A price is paid for this simplicity and generality in that the Frechet bounds of correlation for the bivariate distributions specified by the marginal distribution given by Moran (1967) are not always attained. Also there is some discontinuity in the bivariate distribution. However this discontinuity can be decreased by giving some distribution to the truncation point over its range for given ρ. Thus the mixture-truncation method is very attractive for simulation studies involving only partly specified dependency structures. The mixture-truncation method may be extended to generate bivariate random vectors having negative values. Another extension may be made to use grade correlation or rank correlation which are invariant under transformation instead of using the product moment correlation.

```
C
C
C
C     THE PROGRAM FOR GENERATING BIVARIATE RADOM VECTCRS(Y,Z)
C     HAVING IDENTICAL MARGINAL DISTRIBUTICN BY MIXTURE-TRUNCA
C      C-TICN METHOD.
C     THE CCRRELATION LIMITS ARE AS FOLLCWS FOR GIVEN MARGINAL
C     DISTRIBUTION.
C        1.UNIFORM: -0.75 < CR < 0.75
C        2.EXPONENTIAL: -0.48 < CR < 0.64
C        3.GAMMA(2,1): -.55 < CR < 0.64
C     GENERATING PROCEDURE
C        CALL BUNF(CR,NV,Y,Z,IS)
C        CALL BEXP(CR,NV,Y,Z,IS)
C        CALL BGAM2(CR,NV,Y,Z,IS)
C     WHERE
C        CR: GIVEN CORRELATICN.
C        NV: NUMBER OF RANDOM VECTORS BE GENERATING.
C        Y: THE FIRST ELEMENT OF VECTOR WITH DIMENSION NV.
C        Z: THE SECOND ELEMENT OF VECTOR WITH DIMENSION NV.
C        IS: INITIAL SEED FOR UNIVARIATE GENERATOR.
C
C
C
C
C
C
C
      SUBRCUTINE BUNF(CR,NV,Y,Z,IU)
C
C
      DIMENSION Y(NV),Z(NV)
C
      IF(CR.GT.0.75 .OR. CR.LT.-0.75) GO TO 70
      IF(CR.LT.0.0) GO TO 5
      XL=0.5-(((9.0-12.0*CR)**0.5)/6.0)
      XU=1.0-XL
      GO TO 10
    5 XL=((-1.0*CR)/3.0)**0.5
      XU=1.0-XL
   10 DO 100 I=1,NV
      CALL RANDOM(IU,U,1)
      XO=XL+(XU-XL)*U
      CALL UPJOB(CR,XO,AP1,AP2,PIE1,PIE2,BETA)
C---CHOOSE TYPE FOR Y
      CALL RANDOM(IU,U,1)
      IF(U.LE.PIE1) GO TO 50
      U=(U-PIE1)/(1.0-PIE1)
      AP=1.0-AP2
C    Y FROM X2
   20 CALL RANDOM(IU,W,1)
      Y(I)=XO+W*(1.0-XO)
C    CHOOSE TYPE FOR Z
      IF(U.LE.AP) GO TO 40
C...Z FROM X2
   30 CALL RANDOM(IU,W,1)
      Z(I)=XO+W*(1.0-XO)
      GO TO 100
C...Z FROM X1
   40 CALL RANDOM(IU,W,1)
      Z(I)=XO*W
      GO TO 100
C....Y FROM X1
   50 U=U/PIE1
   60 CALL RANDOM(IU,W,1)
      Y(I)=XO*W
      IF(U.LE.AP1) GO TO 40
      GO TO 30
  100 CONTINUE
      RETURN
   70 WRITE(6,80)CR
   80 FORMAT(5X,'CHECK CORRELATION LIMIT,DO NOT ALLOW GIVEN,
```

```
     CCORRELATION IN THIS PROGRAM FOR CR=',F7.3)
      STOP
      END
C
C
      SUBROUTINE UPJOB(CR,XO,AP1,AP2,PIE1,PIE2,BETA)
C---UNIFORM
C---FIND INITIAL VALUE
      AP1=(CR+3.0*XO**2)/(3.0*XO)
      AP2=1.0-(1.0-AP1)*(XO/(1.0-XO))
      PIE1=(1.0-AP2)/(1.0-AP1+1.0-AP2)
      PIE2=(1.0-AP1)/(1.0-AP1+1.0-AP2)
      BETA=AP1-(1.0-AP2)
      RETURN
      END
C
C
C
C
C
      SUBROUTINE BEXP(CR,NV,Y,Z,IS)
C
C
      DIMENSION Y(NV),Z(NV)
C
      IX=IS*0.8
      IU=IS*0.3+123456
      IF(CR.GT.0.64 .OR. CR.LT.-0.48) GO TO 70
      CALL BOUND(CR,XS1,XS2,XL,XU)
      DO 100 I=1,NV
      CALL RANDOM(IU,U,1)
      XO=XL+(XU-XL)*U
      CALL EPJOB(CR,XO,AP1,AP2,PIE1,PIE2,BETA)
C---CHOOSE TYPE FOR Y
      CALL RANDOM(IU,U,1)
      IF(U.LE.PIE1) GO TO 50
      U=(U-PIE1)/(1.0-PIE1)
      AP=1.0-AP2
C---Y FROMX2
   20 CALL EXPON(IX,X,1)
      Y(I)=X+XO
C---CHOOSE TYPE FOR Z
      IF(U.LE.AP) GO TO 40
C---Z FROM X2
   30 CALL EXPON(IX,X,1)
      Z(I)=X+XO
      GO TO 100
C---Z FROM X1
   40 CALL RANDOM(IU,U,1)
      Z(I)=-1.0*ALOG(1.0-U*PIE1)
      GO TO 100
C---Y FROM X1
   50 U=U/PIE1
   60 CALL RANDOM(IU,U,1)
      Y(I)=-1.0*ALOG(1.0-U*PIE1)
      IF(U.LE.AP1) GO TO 40
      GO TO 30
  100 CONTINUE
      RETURN
   70 WRITE(6,80)CR
   80 FORMAT(5X,'CHECK CORRELATION LIMIT,DO NOT ALLOW GIVEN,
     CCORRELATION IN THIS PROGRAM FOR CR=',F7.3)
      STOP
      END
C
C
      SUBROUTINE EPJOB(CR,XO,AP1,AP2,PIE1,PIE2,BETA)
C    EXPONENTIAL
      PIE1=1.0-(1.0/EXP(XO))
      PIE2=1.0-PIE1
      AP1=PIE1*(1.0+(CR/XO**2))
      AP2=1.0-((PIE1/PIE2)*(1.0-AP1))
```

```
      BETA=AP1-(1.0-AP2)
      RETURN
      END
C
C
      SUBROUTINE BOUND(RO,XS1,XS2,XL,XU)
      IF(RO.LT.0.0) GO TO 10
      AA=(1.0-RO-((RO**2)*(5.0/12.0)))**0.5
      XS1=((1.0-0.5*RO)-AA)/(RO/3.0)
      XS2=((1.0-0.5*RO)+AA)/(RO/3.0)
      CALL RPSON(XS1,XX,RO)
      XL=XX
      CALL RPSON(XS2,XX,RO)
      XU=XX
      RETURN
   10 AR=ABS(RO)
      XL=AR**0.5
      XS1=0.0
      XS2=((AR**0.5)-AR)/(AR*0.5)
      CALL RPSON(XS2,XX,RO)
      XU=XX
      RETURN
      END
C
C
      SUBROUTINE RPSON(XS,XX,RO)
      EE=0.0001
      Y1=XS
      N=1
   10 IF(RO.LT.0.0) GO TO 20
      CALL FUNP(Y1,FV,DV,RO)
      GO TO 25
   20 CALL FUNN(Y1,FV,DV,RO)
   25 Y2=Y1-(FV/DV)
      IF((ABS(Y1-Y2)).LE.EE) GO TO  30
      Y1=Y2
      GO TO 10
   30 XX=Y2
      RETURN
      END
C
C
      SUBROUTINE FUNP(Y1,FV,DV,RO)
      FV=(Y1**2)-(RO*(EXP(Y1)-1.0))
      DV=(2.0*Y1)-(RO*EXP(Y1))
      RETURN
      END
C
C
      SUBROUTINE FUNN(Y1,FV,DV,RO)
      RR=ABS(RO)
      FV=(Y1**2)-RR*((EXP(Y1)-1.0)**2)
      DV=2.0*Y1-(2.0*RR*EXP(2.0*Y1))+(2.0*RR*EXP(Y1))
      RETURN
      END
C
C
C
C
C
      SUBROUTINE BGAM2(CR,NV,Y,Z,IS)
C
C
      DIMENSION Y(NV),Z(NV)
C
      IX=IS*0.8
      IU=IS*0.3+123456
      IF(CR.GT.0.64 .OR. CR.LT.-0.55) GO TO 70
      CALL GROUND(CR,XS1,XS2,XL,XU)
      DO 100 I=1,NV
      CALL RANDOM(IU,U,1)
      X0=XL+(XU-XL)*U
```

```
      CALL GPJOB(CR,XO,AP1,AP2,PIE1,PIE2,BETA)
C---CHOOSE TYPE FOR Y
      CALL RANDCM(IU,U,1)
      IF(U.LE.PIE1) GO TO 50
      U=(U-PIE1)/(1.0-PIE1)
      AP=1.0-AP2
C---Y FROMX2
   20 CALL  GAMA(2.0,IX,X,1)
      IF(X.LE.XO) GO TO 20
      Y(I)=X
C---CHOOSE TYPE FOR Z
      IF(U.LE.AP) GO TO 40
C---Z FROM X2
   30 CALL GAMA(2.0,IX,X,1)
      IF(X.LE.XO) GO TO 30
      Z(I)=X
      GO TO 100
C---Z FROM X1
   40 CALL GAMA(2.0,IX,X,1)
      IF(X.GT.XO) GO TO 40
      Z(I)=X
      GO TO 100
C---Y FROM X1
   50 U=U/PIE1
   60 CALL GAMA(2.0,IX,X,1)
      IF(X.GT.XO) GO TO 60
      Y(I)=X
      IF(U.LE.AP1) GO TO 40
      GO TO 30
  100 CONTINUE
      RETURN
   70 WRITE(6,80)CR
   80 FORMAT(5X,'CHECK CORRELATION LIMIT,DO NOT ALLOW GIVEN,
     CCORRELATION IN THIS PROGRAM FOR CR=',F7.3)
      STOP
      END
C
C
      SUBROUTINE GPJOB(CR,XO,AP1,AP2,PIE1,PIE2,BETA)
C    GAMMA(2,1)
      EX=1.0/EXP(XO)
      PIE1=1.0-EX-XO*EX
      PIE2=1.0-PIE1
      AD=(XO**4)*(1.0/EXP(2.0*XO))
      AP1=PIE1+((2.0*CR*PIE1*PIE2**2)/AD)
      AP2=1.0-((PIE1/PIE2)*(1.0-AP1))
      BETA=AP1-(1.0-AP2)
      RETURN
      END
C
C
      SUBROUTINE GBOUND(RO,XS1,XS2,XL,XU)
      IF(RO.LT.0.0) GO TO 10
      A1=((4.0*RO)/3.0)+SQRT(((4.0*RO**2)/9.0)+(4.0*RO))
      A2=2.0*(1.0-(RO/3.0))
      XS1=A1/A2
      XS1=XS1+1.0
      XS2=XS1+10.0
      CALL GRPSON(XS1,XX,RO)
      XL=XX
      CALL GRPSON(XS2,XX,RO)
      XU=XX
      RETURN
   10 AR=2.0*ABS(RO)
      XL=(SQRT(AR)+SQRT(AR+SQRT(AR)*4.0))/2.0
      XS1=0.0
      A1=SQRT((SQRT(AR)/6.0)+(RO/9.0))-(SQRT(AR)/6.0)
      A2=SQRT(AR)/12.0
      XS2=A1/A2
      CALL GRPSON(XS2,XX,RO)
      XU=XX
      RETURN
```

```
      END
C
C
      SUBROUTINE GRPSON(XS,XX,RO)
      EE=0.001
      Y1=XS
   10 IF(RO.LT.0.0) GO TO 20
      CALL GFUNP(Y1,FV,DV,RO)
      GO TO 25
   20 CALL GFUNN(Y1,FV,DV,RO)
   25 Y2=Y1-(FV/DV)
      IF((ABS(Y1-Y2)).LE.EE) GO TO  30
      Y1=Y2
      GO TO 10
   30 XX=Y2
      RETURN
      END
C
C
      SUBROUTINE GFUNP(Y1,FV,DV,RO)
      F1=(Y1**4)+(2.0*RO*(1.0+Y1)**2.0)
      F2=2.0*RO*EXP(Y1)*(1.0+Y1)
      FV=F1-F2
      DF1=(4.0*Y1**3)+(4.0*RO*(1.0+Y1))
      DF2=2.0*RO*EXP(Y1)*(2.0+Y1)
      DV=DF1-DF2
      RETURN
      END
C
C
      SUBROUTINE GFUNN(Y1,FV,DV,RO)
      TR=SQRT(2.0*ABS(RO))
      F1=(Y1**2.0)+(TR*(1.0+Y1))
      F2=TR*EXP(Y1)
      FV=F1-F2
      DF1=(2.0*Y1)+TR
      DF2=TR*EXP(Y1)
      DV=DF1-DF2
      RETURN
      END
C
C
C
C
C
      SUBROUTINE  BIVHST  (X, Y, N, WORK)
C
      IMPLICIT REAL*8 (D)
      INTEGER*2  CELL, MAX, SCALE, ONE, ONE5
      INTEGER    RUNS, WN, SN, WEIGHT, U, FLAG
      REAL       D, DELTA, DELTAX
C
      DIMENSION X(N), Y(N), WORK(N)
      DIMENSION XMOM(6), YMOM(6), DWORK(8), CELL(32,32)
      DIMENSION KEY(16),XLABEL(8)
      DATA      NOUT/6/, ONE/1/, ONE5/15/
C
      IF(N .GT. 15) GO TO 20
      WRITE(NOUT,10) N
   10 FORMAT('1***BIVHST***  TOO FEW SAMPLE POINTS. N =',I9)
      RETURN
C
   20 AN = N
      AN2 = AN * AN
      NM = N - 1
C-----ORDER X-SAMPLE AND FIND RANGES
      CALL SORTON (X,1,N,Y)
      XRANGE = X(N) - X(1)
      YMAX = Y(1)
      YMIN = Y(1)
      DO 30 I=2,N
      IF(Y(I) .GT. YMAX) YMAX = Y(I)
```

```
      IF(Y(I) .LT. YMIN) YMIN = Y(I)
   30 CONTINUE
      YRANGE = YMAX - YMIN
      IF(XRANGE * YRANGE .GT. 0.) GO TO 50
      WRITE(NOUT,40)
   40 FORMAT('1***BIVHST***  X OR Y SAMPLE HAS ZERO RANGE')
      RETURN
C
C-----ZERO OUT DATA ARRAY
   50 DO 70 I=1,32
      DO 60 J=1,32
      CELL(I,J) = 0
   60 CONTINUE
   70 CONTINUE
C-----FIND SCALE FACTORS
      AX = 31.999 / XRANGE
      BX = 1.0001 - AX * X(1)
      DELTAX = XRANGE / 32.
      AY = 31.999 / YRANGE
      BY = 1.0001 - AY * YMIN
      DELTA = YRANGE / 32.
      MAX = 0
C-----ACCUMULATE CELL COUNTS
      DO 100 I=1,N
      IX = AX * X(I) + BX
      IY = AY * Y(I) + BY
      IY = 33 - IY
      CELL(IX, IY) = CELL(IX, IY) + ONE
      IF(CELL(IX, IY) .GT. MAX) MAX = CELL(IX, IY)
  100 CONTINUE
C-----SCALE FACTOR FOR COUNTS
      SCALE = (MAX - ONE) / ONE5 + ONE
C-----FIND SAMPLE MOMENTS
      DWORK(1) = 0.0
      CALL ACCUM(X,N,DWORK)
      CALL MOMENT(DWORK, XMOM)
      DWORK(1) = 0.0
      CALL ACCUM(Y,N,DWORK)
      CALL MOMENT(DWORK, YMOM)
C-----FIND COVARIANCE
      DSUM = 0.0
      DO 110 I=1,N
      DSUM = DSUM + (X(I) - XMOM(1)) * (Y(I) - YMOM(1))
  110 CONTINUE
      COVAR = DSUM / AN
      XDEV = SQRT(XMOM(2))
      YDEV = SQRT(YMOM(2))
      RHO = COVAR / (XDEV * YDEV)
C-----KENDALL'S TAU COEFFICIENT
      T = 0.0
      IF(N .GT. 500) GO TO 119
      DO 118 I=1,NM
      DO 114 J=I,N
      Z = Y(J) - Y(I)
      IF(Z .GT. 0.) T = T + 2.0
      IF(Z .LT. 0.) T = T - 2.0
  114 CONTINUE
  118 CONTINUE
      T = T / (AN2 - AN)
C-----SET UP WORK AREA AND ORDER Y SAMPLE
  119 CONTINUE
      WI = 1.0
      DO 120 I=1,N
      WORK(I) = WI
      WI = WI + 1.0
  120 CONTINUE
      CALL SORTON(Y, 1, N, WORK)
C-----FIND RANK-ORDER STATISTICS
      WN = 0
      SN = 0
      RUNS = 0
      IF(Y(1) .LT. X(1)) RUNS = 1
```

```
      FLAG = 0
      C = 0.0
      I = 1
      IX = 1
      IY = 1
  130 Z = ABS(FLOAT(IX-IY)) / AN
      IF(C .LT. Z) C = Z
      IF(X(IX) - Y(IY)) 150, 160, 140
C-----CURRENT VALUE IN POOLED SAMPLE IS A Y
  140 RUNS = RUNS + FLAG
      FLAG = 0
      IY = IY + 1
      I = I + 1
      IF(IY .GT. N) GO TO 200
      GO TO 130
C-----CURRENT VALUE IN POOLED SAMPLE IS AN X
  150 RUNS = RUNS + 1 - FLAG
      FLAG = 1
      WN = WN + I
      SN = SN + WEIGHT(I,N)
      IX = IX + 1
      IF(IX .GT. N) GO TO 220
      I = I + 1
      GO TO 130
C-----X AND Y VALUES ARE TIED. USE MIDRANK CORRECTION.
  160 BASE = X(IX)
      M = 1
      J1 = I + I + 1
      J2 = WEIGHT(I,N) + WEIGHT(I+1,N)
      K = I
      I = I + 1
C-----GET TIED X VALUES
  170 IX = IX + 1
      IF(IX .GT. N) GO TO 180
      IF(X(IX) .NE. BASE) GO TO 180
      M = M + 1
      I = I + 1
      J1 = J1 + 1
      J2 = J2 + WEIGHT(I,N)
      GO TO 170
C-----FIND TIED Y VALUES
  180 IY = IY + 1
      IF(IY .GT. N)GO TO 190
      IF(Y(IY) .NE. BASE) GO TO 190
      I = I + 1
      J1 = J1 + I
      J2 = J2 + WEIGHT(I,N)
      GO TO 180
C-----WEIGHTED AVERAGE OF TIED VALUES
  190 L = I - K + 1
      Z = FLOAT(M * J1) / FLOAT(L) + 0.5
      WN = WN + Z
      Z = FLOAT(M * J2) / FLOAT(L) + 0.5
      SN = SN + Z
C-----AD HOC CORRECTION FOR RUNS
      RUNS = RUNS + FLOAT(2 * M * (L-M)) / FLOAT(L)
      I = I + 1
      IF(IX .GT. N) GO TO 220
      IF(IY .LE. N) GO TO 130
C-----Y OBSERVATIONS EXHAUSTED
  200 RUNS = RUNS + 1
      N2 = N + N
      DO 210 I=I,N2
      WN = WN + I
      SN = SN + WEIGHT(I,N)
  210 CONTINUE
C-----FIND MANN-WHITNEY STATISTIC
  220 U = WN - (N * (N + 1)) / 2
C-----SPEARMAN'S RANK CORRELATION COEFFICIENT
      DSUM = 0.
      WI = 1.0
      DO 230 I=1,N
```

```
      DSUM = DSUM + (WI - WORK(I)) ** 2
      WI = WI + 1.0
  230 CONTINUE
      R = 1. - 6. * DSUM / (AN * (AN2 - 1.))
C-----FIND SAMPLE MEDIANS
      IF(MOD(N,2) .EQ. 0) GO TO 240
      M = 1 + N/2
      XMED = X(M)
      YMED = Y(M)
      GO TO 250
  240 M = N/2
      XMED = 0.5 * (X(M) + X(M+1))
      YMED = 0.5 * (Y(M) + Y(M+1))
C-----FIND NORMALIZED STATISTICS
  250 RNORM = (RUNS - N - 1) * SQRT((AN + AN - 1.) / (AN2 -
     CAN))
      FAC = SQRT(12. / (AN2 * (AN + AN + 1.)))
      UNORM = (U - 0.5 * AN2) * FAC
      WNORM = (WN - AN2 - 0.5 * AN) * FAC
      SNORM = (SN - AN2 - 0.5 * AN) * FAC
C-----REORDER Y SAMPLE
      CALL SORTON (WORK, 1, N, Y)
C
C     WRITE OUTPUT
C
C-----HEADING
      WRITE(NOUT, 300) N
  300 FORMAT('1',20X,'BIVARIATE SAMPLE SUMMARY',10X,'SAMPLE
     C SIZE =',I9//)
      WRITE(NOUT,310)
  310 FORMAT(13X,'+--',7('*',7('-')),'*-----+')
C
C-----BODY OF PLOT
      CALL OUTPUT(CELL(1, 1), SCALE, 1, 0.)
      CALL REPEAT
      CALL OUTPUT(CELL(1, 2), SCALE,  2, YMAX -      DELTA)
      WRITE(NOUT,320)
  320 FORMAT('+',93X,'MEASURES OF ASSOCIATION')
      CALL REPEAT
      CALL OUTPUT(CELL(1, 3), SCALE, 3, 0.)
      WRITE(NOUT,330) COVAR
  330 FORMAT('+',83X,'COVARIANCE',25X,1PE14.6)
      CALL REPEAT
      WRITE(NOUT,340) RHO
  340 FORMAT('+',83X,'CORRELATION COEFFICIENT',13XF9.6)
      CALL OUTPUT(CELL(1, 4), SCALE, 4, 0.)
      WRITE(NOUT,350) R
  350 FORMAT('+',83X,'SPEARMAN''S RANK CORRELATION COEF.   '
     C,F9.6)
      CALL REPEAT
      IF ( N .LE. 500 ) WRITE(NOUT,360) T
  360 FORMAT('+',83X,'KENDALL''S TAU COEFFICIENT',11XF9.6)
      CALL OUTPUT(CELL(1, 5), SCALE, 5, 0.)
      CALL REPEAT
      CALL OUTPUT(CELL(1, 6), SCALE,  6, YMAX - 5. * DELTA)
      WRITE(NOUT,370)
  370 FORMAT('+',95X,'TESTS FOR EQUIDISTRIBUTION')
      CALL REPEAT
      CALL OUTPUT(CELL(1, 7), SCALE, 7, 0.)
      WRITE(NOUT,380)
  380 FORMAT('+',111X,'TEST',7X,'NORMALIZED')
      CALL REPEAT
      WRITE(NOUT,390)
  390 FORMAT('+',109X,'STATISTIC',5X,'STATISTIC')
      CALL OUTPUT(CELL(1, 8), SCALE, 8, 0.)
      WRITE(NOUT,400) D
  400 FORMAT('+',83X,'KOLMOGOROV-SMIRNOV TEST ',F10.6)
      CALL REPEAT
      WRITE(NOUT,410) RUNS, RNORM
  410 FORMAT('+',83X,'WALD-WOLFOWITZ RUNS TEST',I10,5XF9.5)
      CALL OUTPUT(CELL(1, 9), SCALE, 9, 0.)
      WRITE(NOUT,420) U, UNORM
```

```
420  FORMAT('+',83X,'MANN-WHITNEY TEST',7XI10,5XF9.5)
     CALL REPEAT
     WRITE(NOUT,430) WN, WNORM
430  FORMAT('+',83X,'WILCOXSON TEST',10XI10,5XF9.5)
     CALL OUTPUT(CELL(1,10), SCALE, 10, YMAX - 9. * DELTA)
     WRITE(NOUT,440) SN, SNORM
440  FORMAT('+',83X,'SIEGEL-TUKEY TEST',7XI10,5XF9.5)
     CALL REPEAT
     CALL OUTPUT(CELL(1,11), SCALE,11, 0.)
     CALL REPEAT
     WRITE(NOUT,450)
450  FORMAT('+',92X,'UNIVARIATE STATISTICS')
     CALL OUTPUT(CELL(1,12), SCALE,12, 0.)
     CALL REPEAT
     WRITE(NOUT,460)
460  FORMAT('+',95X,'X SAMPLE',9X,'Y SAMPLE')
     CALL OUTPUT(CELL(1,13), SCALE,13, 0.)
     WRITE(NOUT,470) XMOM(1), YMOM(1)
470  FORMAT('+',83X,'MEAN',5X1PE14.6,3XE14.6)
     CALL REPEAT
     WRITE(NOUT,480) XMED, YMED
480  FORMAT('+',83X,'MEDIAN',3X1PE14.6,3XE14.6)
     CALL OUTPUT(CELL(1,14), SCALE, 14, YMAX -13. * DELTA)
     CALL REPEAT
     WRITE(NOUT,490) XMOM(2), YMOM(2)
490  FORMAT('+',83X,'VARIANCE',1X1PE14.6,3XE14.6)
     CALL OUTPUT(CELL(1,15), SCALE,15, 0.)
     WRITE(NOUT,500) XDEV, YDEV
500  FORMAT('+',83X,'STD DEV',2X1PE14.6,3XE14.6)
     CALL REPEAT
     WRITE(NOUT,510) XRANGE, YRANGE
510  FORMAT('+',83X,'RANGE',4X1PE14.6,3XE14.6)
     CALL OUTPUT(CELL(1,16), SCALE,16, 0.)
     CALL REPEAT
     WRITE(NOUT,520) XMOM(4), YMOM(4)
520  FORMAT('+',83X,'SKEWNESS',1X1PE14.6,3XE14.6)
     CALL OUTPUT(CELL(1,17), SCALE,17, 0.)
     WRITE(NOUT,530) XMOM(6), YMOM(6)
530  FORMAT('+',83X,'KURTOSIS',1X1PE14.6,3XE14.6)
     CALL REPEAT
     CALL OUTPUT(CELL(1,18), SCALE, 18, YMAX -17. * DELTA)
     WRITE(NOUT,540) X(N), YMAX
540  FORMAT('+',83X,'MAXIMUM',2X1PE14.6,3XE14.6)
     CALL REPEAT
     WRITE(NOUT,550) X(1), YMIN
550  FORMAT('+',83X,'MINIMUM',2X1PE14.6,3XE14.6)
     Z = YMAX - 17. * DELTA
     DO 560 J = 19,32
     Z = Z - DELTA
     CALL OUTPUT(CELL(1,J), SCALE, J, Z)
     CALL REPEAT
560  CONTINUE
     WRITE(NOUT,310)
     XLABEL(1) = X(1) + DELTAX
     FOURD = 4. * DELTAX
     DO 570 I=2,8
     XLABEL(I) = XLABEL(I-1) + FOURD
570  CONTINUE
     WRITE(NOUT,580)(XLABEL(I),I=1,7,2),(XLABEL(J),J=2,8,2)
580  FORMAT(16X,8('|',7X)/ 12X,4(1PE10.3,'   |     ')/ 20X,4(E
    C10.3,6X))
     KEY(1) = 0
     DO 590 I=2,16
     KEY(I) = SCALE * I - 1
590  CONTINUE
     WRITE(NOUT,600) (KEY(I),I=1,16)
600  FORMAT(//50X,'KEY',//' SYMBOL PRINTED',12X,'-       =
    1 -       =       =       =       =       T       >       T     A
    2 F       H       F       F'/  '+',38X,'.        .       I       *
    3   +       X       <       X       V       E       E       E       E'
    4/  '+',56X,'.        .       -       -       T       =
    5/      T           X'/ '+',110X,'T'//
```

```
     6   ' NO. OBSERVATIONS',I5,15I6)
         RETURN
         END
         FUNCTION WEIGHT(I, N)
         INTEGER WEIGHT
C
C        THIS FUNCTION SUBPROGRAM COMPUTES THE THE WEIGHT FUNCT
C        ION FOR THE I-TH RANK IN A SIEGEL-TUKEY TEST WITH A SA
C        -MPLE OF N.
C
         K = MOD(I,2)
         WEIGHT = I + I - K
         IF(I .GT. N) WEIGHT = 4 * N - WEIGHT + 2 * (1 - K)
         RETURN
         END
         SUBROUTINE OUTPUT (COUNTS, SCALE, L, Y)
C
C        THIS SUBROUTINE OUTPUTS A SINGLE LINE OF THE BIVARIATE
C        HISTOGRAM ON THE LINE PRINTER
C        ON THE LINE PRINTER
C
         INTEGER*2  COUNTS, SCALE, ZERO, ONE
         LOGICAL*1  LINE, CHAR
         DIMENSION  LINE(64,4), CHAR(4, 16), COUNTS(32), N(16),
        CCHA(16)
         EQUIVALENCE (CHAR(1,1), CHA(1))
         DATA       ONE/1/, ZERO/0/, NOUT/6/, N/3*1,3*2,2*3,2,6
        C*3,4/
         DATA       CHA /'    ','-   ','=   ','-.  ','=.  ','=I
        1   ','=*. ',                '=+. ','TX  ','><- ','TX-
        2','AVT ','HE= ','HE/ ',              'HET ','HEXT'/
C
         LINES = 1
         K = 1
         DO 20 J=1,32
         INDEX = COUNTS(J) / SCALE + ONE
         IF(INDEX .EQ. 1 .AND. COUNTS(J) .GT. ZERO) INDEX = 2
         IF(N(INDEX) .GT. LINES) LINES = N(INDEX)
         DO 10 I=1,4
         LINE(K,I) = CHAR(I,INDEX)
         LINE(K+1,I) = CHAR(I,INDEX)
   10    CONTINUE
         K = K + 2
   20    CONTINUE
         IF(MOD(L-2,4) .EQ. 0) GO TO 40
         ENTRY REPEAT
         WRITE(NOUT,30) ((LINE(J,I), J=1,64), I=1,LINES)
   30    FORMAT(13X,'|',64A1,'|'/('+',13X,64A1))
         GO TO 60
   40    WRITE(NOUT,50) Y, ((LINE(J,I), J=1,64), I=1,LINES)
   50    FORMAT(1X1PE10.3,'  *',64A1,'*'/ ('+',13X,64A1))
   60    RETURN
         END
C
C
         SUBROUTINE MOMENT (WORK, XMOM)
C
C        PURPOSE:
C           FIND THE MOMENTS OF THE DATA ARRAY X WHEN NOT ALL
C        OF THE ARRAY IS AVAILABLE AT ANY GIVEN TIME.
C
C        USAGE:
C           DATA POINTS ARE PASSED TO THE SUBROUTINE "M" AT A T
C        -IME WITH THE RUNNING TOTALS KEPT IN THE DOUBLE PRECIS
C        -ION WORK AREA VECTOR 'WORK'.  PRIOR TO THE FIRST CALL
C        ,WORK(1) SHOULD BE SET TO 0.0. DATA IS THEN PASSED BY
C        THE CALL
C                CALL ACCUM(X, M, WORK)
C        AN INITIAL MEAN ESTIMATE IS KEPT IN WORK(4) AFTER THE
C        FIRST 7 DATA POINTS HAVE BEEN PASSED.  HIGHER CENTRAL
C        MOMENTS ARE FOUND USING THIS ESTIMATE. ONCE AT LEAST 7
C        X VALUES HAVE BEEN ACCUMULATED, RESULTS MAY BE OBTAINE
```

```
C      -D FROM
C              CALL MCMENT(WORK, XMOM)
C      RESULTS ARE RETURNED AS FOLLOWS:
C         XMOM(1)   THE SAMPLE MEAN
C         XMOM(2)   THE SAMPLE VARIANCE (UNBIASED ESTIMATOR)
C         XMOM(3)   THE SAMPLE THIRD CENTRAL MOMENT (UNBIASED
C                   ESTIMATOR)
C         XMOM(4)   THE SAMPLE COEFFICIENT OF SKEWNESS
C         XMOM(5)   THE SAMPLE FOURTH CENTRAL MOMENT (UNBIASED
C                   ESTIMATOR)
C         XMOM(6)   THE SAMPLE COEFFICIENT OF KURTOSIS
C
C
      IMPLICIT  REAL*8 (D)
      REAL*8     WORK
      DIMENSION  WORK(8), XMOM(6), X(M)
C
      IF(WORK(1) .LT. 7.0D0) RETURN
C-----FIND CORRECTION FACTORS
      DMEAN = (WORK(2) + WORK(3)) / WORK(1)
      DC = DMEAN - WORK(4)
      DC2 = DC * DC * WORK(1)
C-----DETERMINE CORRECT CENTRAL MOMENTS
      DM2 = WORK(5) - DC2
      DM3 = WORK(6) + WORK(7) - DC * (3.D0 * WORK(5) - DC2 -
     CDC2)
      DM4 = WORK(8) - DC * (4.D0 * (WORK(6) + WORK(7)) - DC
     1          *(6.D0 * WORK(5) - 3.D0 * DC2))
C-----CORRECT ESTIMATORS IN WORK ARRAY
      WORK(4) = DMEAN
      WORK(5) = DM2
      WORK(6) = 0.
      WORK(7) = 0.
      IF(DM3 .GT. 0.) WORK(6) = DM3
      IF(DM3 .LT. 0.) WORK(7) = DM3
      WORK(8) = DM4
C-----RETURN STATISTICS
      XMOM(1) = DMEAN
      DFAC = WORK(1) - 1.D0
      XMOM(2) = DM2 / DFAC
      DFAC = DFAC * (WORK(1) - 2.D0)
      XMOM(3) = WORK(1) * DM3 / DFAC
      XMOM(4) = XMOM(3) / (XMOM(2) * SQRT(XMOM(2)))
      DFAC = DFAC * (WORK(1) - 3.D0)
      XMOM(5) = (((WORK(1) - 2.D0) * WORK(1) + 3.D0) * DM4
     1          - (6.D0 * WORK(1) - 9.D0) * DM2 * DM2/
     2WORK(1)) /DFAC
      XMOM(6) = XMOM(5) / (XMOM(2) * XMOM(2)) - 3.0
      RETURN
C
C
      ENTRY ACCUM(X, M, WORK)
C
      IF(WORK(1) .GT. 6.D0) GO TO 210
C-----ACCUMULATE FIRST 7 DATA POINTS
      N = WORK(1) + 2.D0
      WORK(1) = WORK(1) + M
      NP = N + M - 1
      IF(NP .GT. 8) NP = 8
      J = 1
      DO 201 I=N, NP
      WORK(I) = X(J)
      J = J + 1
  201 CONTINUE
      IF(NP .LT. 8) RETURN
C-----OBTAIN INITIAL MEAN ESTIMATE
      DSUMP = 0.
      DSUMM = 0.
      DO 202 I = 2,8
      IF(WORK(I) .LT. 0.) DSUMM = DSUMM + WORK(I)
      IF(WORK(I) .GT. 0.) DSUMP = DSUMP + WORK(I)
  202 CONTINUE
```

```
      IF(J .GT. M) GO TO 204
      DO 203 I=J,M
      IF(X(I) .LT. 0.) DSUMM = DSUMM + X(I)
      IF(X(I) .GT. 0.) DSUMP = DSUMP + X(I)
  203 CONTINUE
  204 DMEAN = (DSUMM + DSUMP) / WORK(1)
C-----SET UP WORK AREA
      D2 = 0.
      D3M = 0.
      D3P = 0.
      D4 = 0.
      DO 205 I=2,8
      DIFF = WORK(I) - DMEAN
      DF2 = DIFF * DIFF
      D2 = D2 + DF2
      IF(DIFF .LT. 0.) D3M = D3M + DIFF * DF2
      IF(DIFF .GT. 0.) D3P = D3P + DIFF * DF2
      D4 = D4 + DF2 * DF2
  205 CONTINUE
      IF(J .GT. M) GO TO 207
      DO 206 I=J,M
      DIFF = X(I) - DMEAN
      DF2 = DIFF * DIFF
      D2 = D2 + DF2
      IF(DIFF .LT. 0.) D3M = D3M + DIFF * DF2
      IF(DIFF .GT. 0.) D3P = D3P + DIFF * DF2
      D4 = D4 + DF2 * DF2
  206 CONTINUE
  207 WORK(2) = DSUMP
      WORK(3) = DSUMM
      WORK(4) = DMEAN
      WORK(5) = D2
      WORK(6) = D3P
      WORK(7) = D3M
      WORK(8) = D4
      RETURN
C-----ACCUMULATE DATA
  210 WORK(1) = WORK(1) + M
      DO 211 I=1,M
      IF(X(I) .GT. 0.) WORK(2) = WORK(2) + X(I)
      IF(X(I) .LT. 0.) WORK(3) = WORK(3) + X(I)
      DIFF = X(I) - WORK(4)
      DF2 = DIFF * DIFF
      WORK(5) = WORK(5) + DF2
      IF(DIFF .GT. 0.) WORK(6) = WORK(6) + DIFF * DF2
      IF(DIFF .LT. 0.) WORK(7) = WORK(7) + DIFF * DF2
      WORK(8) = WORK(8) + DF2 * DF2
  211 CONTINUE
      RETURN
      END
C
C
      SUBROUTINE  SORTON (ON, II, JJ, WITH)
C
C     THIS SUBROUTINE SORTS THE VECTOR ON INTO
C     ASCENDING ORDER FROM ON(II) TO ON(JJ), CARRYING
C     ALONG THE CORRESPONDING ELEMENTS OF THE VECTOR
C     WITH.
C
      INTEGER HI, HISTK, HIKEEP, STKPT, WITH
      DIMENSION  ON(JJ), WITH(JJ), HISTK(16), LOSTK(16)
C
      STKPT = 1
      LOKEEP = II
      HIKEEP = JJ
   10 IF (LOKEEP .GE. HIKEEP)  GO TO 100
   20 LO = LOKEEP - 1
      HI = HIKEEP + 1
      MIDDLE = (LOKEEP + HIKEEP) / 2
      IL = LOKEEP
      IH = HIKEEP
      IF ( ON(MIDDLE) .GT. ON(HIKEEP) ) IH = MIDDLE
```

```
      IF ( ON(MIDDLE) .LT. ON(LOKEEP) ) IL = MIDDLE
      IF ( ON(HIKEEP) .LT. ON(IL)     ) IL = HIKEEP
      IF ( ON(LOKEEP) .GT. ON(IH)     ) IH = LOKEEP
      ISUB = LOKEEP + MIDDLE + HIKEEP - IH - IL
      TEST = ON(ISUB)
      GO TO 50
C
   30 IF (LO .EQ. HI) GO TO 50
      TEMP   = ON(HI)
      ON(HI) = ON(LO)
      ON(LO) = TEMP
      ITEMP    = WITH(HI)
      WITH(HI) = WITH(LO)
      WITH(LO) = ITEMP
   50 HI = HI - 1
      IF ( ON(HI) .GT. TEST) GO TO 50
   70 LO = LO + 1
      IF ( ON(LO) .LT. TEST) GO TO 70
      IF ( LO .LE. HI ) GO TO 30
      IF(HI - LOKEEP .LE. HIKEEP - LO) GO TO 90
      LOSTK(STKPT) = LOKEEP
      HISTK(STKPT) = HI
      STKPT = STKPT + 1
      LOKEEP = LO
      GO TO 110
   90 LOSTK(STKPT) = LO
      HISTK(STKPT) = HIKEEP
      STKPT = STKPT + 1
      HIKEEP = HI
      GO TO 110
  100 STKPT = STKPT - 1
      IF(STKPT .EQ. 0) RETURN
      LOKEEP = LOSTK(STKPT)
      HIKEEP = HISTK(STKPT)
  110 IF ( HIKEEP - LOKEEP .GE. 11) GO TO 20
      IF ( LOKEEP .EQ. II) GO TO 10
      LOKEEP = LOKEEP - 1
  120 LOKEEP = LOKEEP + 1
      IF ( LOKEEP .EQ. HIKEEP ) GO TO 100
      IF ( ON(LOKEEP) .LE. ON(LOKEEP+1) ) GO TO 120
      TEMP = ON(LOKEEP + 1)
      LO = LOKEEP
  130 ON(LO+1) = ON(LO)
      LO = LO - 1
      IF ( TEMP .LT. ON(LO)) GO TO 130
      ON(LO+1) = TEMP
      IW = LOKEEP + 1
      N = LOKEEP - LO
      ITEMP = WITH(IW)
      DO 140 IMOVE = 1, N
      WITH(IW) = WITH(IW-1)
      IW = IW - 1
  140 CONTINUE
      WITH(IW) = ITEMP
      GO TO 120
      END
```

```
C
C
C     FCR GENERATING EIVARIATE RANDOM VECTORS(Y,Z),CALL
C     FCLLCWING SUBROLTINE.
C       1. UNIFORM(0,1) MARGINAL DISTRIBUTICN.
C          CALL LEWUFM(CR,NV,Y,Z,IS): POSITIVE CORRELATION BY
C          LAWERENCE AND LEWIS METHOD.
C          CALL GAVUFM(CR,NV,Y,Z,IS): NEGATIVE CORRELATION BY
C          TRANSFORMATION OF GAVER'S EXPONENTIAL.
C       2. EXPONENTIAL(MEAN=1.0) MARGINAL DISTRIBUTICN.
C          CALL MAREXP(CR,NV,Y,Z,IS): POSITIVE CORRELATION BY
C          MARSHAL AND OLKINS METHOD.
C          CALL GAVEXP(CR,NV,Y,Z,IS): NEGATIVE CORRELATION BY
C          GAVER'S METHOD.
C     WHERE
C       CR: GIVEN CORRELATICN.
C       NV: NUMBER OF RANDCM VECTORS BE GENERATING.
C       Y: THE FIRST ELEMENT OF VECTOR WITH DIMENSICN NV.
C       Z: THE SECOND ELEMENT CF VECTOR WITH DIMENSION NV.
C       IS: INITIAL RANDOM SEED.
C
C
C
C
      SUBRCUTINE LEWUFM(CR,NV,Y,Z,IU)
C
      DIMENSION Y(NV),Z(NV),U(3)
      IF (CR.LT.0.0) GO TC 20
      A=SQRT(36.0+(12.0*CR)+(16.0*(CR**2.0)))-6.0
      BETA=A/(2.0*CR)
      ALPA=2.0-(1.0/BETA)
      P=(1.0-BETA)/(1.0-BETA+(ALPA*BETA))
      R=1.0-ALPA
      DO 100 I=1,NV
      CALL RANDOM(IU,U,2)
      Y(I)=U(1)
      IF(U(2).LE.R) GO TC 50
      W=(U(2)-R)/(1.0-R)
      IF(W.LE.P) GC TO 10
      Z(I)=(Y(I)**BETA)*((W-P)/(1.0-P))**(R*BETA)
      GC TC 100
   10 Z(I)=(Y(I)**BETA)*(W/P)
      GO TC 100
   50 W=U(2)/R
      IF(W.LE.P) GC TO 60
      Z(I)=((W-P)/(1.0-P))**(R*BETA)
      GC TC 100
   60 Z(I)=W/P
  100 CONTINUE
      RETURN
   20 WRITE(6,30)CR
   30 FORMAT(5X,'THIS METHCD NOT ALLOW CR=',F7.3)
      RETURN
      END
C
C
C
C
      SUBROUTINE GAVUFM(CR,NV,Y,Z,IS)
C
      DIMENSION Y(NV),Z(NV),WK(1),G(100)
      DOUBLE PRECISION DXS
      DXS=IS*0.8
      IU=IS*0.5+12345
      IF (CR.GT.0.0) GO TC 20
      P=-2.0*CR
      Q=1.0-P
      DO 100 I=1,NV
      CALL GGEOT(DXS,1,Q,WK1,IG)
      CALL RANDOM(IU,U,1)
      RN=1.0/IG
      Y(I)=(Q*(U**RN))/(1.0-(P*(U**RN)))
```

```
      CALL RANDOM(IU,G,IG)
      W=1.0
      DO 10 K=1,IG
      W=W*G(K)
   10 CONTINUE
      Z(I)=W**(1.0-P)
  100 CONTINUE
      RETURN
   20 WRITE(6,30)CR
   30 FORMAT(5X,'THIS METHOD NOT ALLOW CR=',F7.3)
      RETURN
      END
C
C
C
C
      SUBROUTINE MAREXP(CR,NV,Y,Z,IX)
C
      DIMENSION Y(NV),Z(NV),E(3)
      IF(CR.LT.0.0) GO TO 20
      R12=(2.0*CR)/(CR+1.0)
      R1=1.0-R12
      R2=R1
      DO 50 I=1,NV
      CALL EXPON(IX,E,3)
      E(1)=E(1)/R1
      E(2)=E(2)/R2
      E(3)=E(3)/R12
      Y(I)=AMIN1(E(1),E(3))
      Z(I)=AMIN1(E(2),E(3))
   50 CONTINUE
      RETURN
   20 WRITE(6,30)CR
   30 FORMAT(5X,'THIS METHOD NOT ALLOW CR=',F7.3)
      RETURN
      END
C
C
C
C
      SUBROUTINE GAVEXP(CR,NV,Y,Z,IS)
C
      DIMENSION Y(NV),Z(NV),WK1(1),WK2(100)
      DOUBLE PRECISION DSEED1,DSEED2,DSEED3
      DSEED1=IS*0.2+123456
      DSEED2=IS*0.5+1234
      DSEED3=IS*0.9+123
      IF(CR.LT.-0.5 .OR. CR.GT.0.0) GO TO 20
      Q=-2.0*CR
      P=1.0-Q
      DO 100 I=1,NV
      CALL GGEOT(DSEED1,1,P,WK1,IG)
      A=IG
      B=P
      CALL GGAMS(DSEED2,A,B,1,WK2,GA)
      Z(I)=GA
      CALL GGUBS(DSEED3,1,U)
      RN=1.0/IG
      Y(I)=ALOG((1.0/(P*U**RN))-(Q/P))
  100 CONTINUE
      RETURN
   20 WRITE(6,30)CR
   30 FORMAT(5X,'THIS METHOD NOT ALLOW CR=',F7.3)
      RETURN
      END
```

BIBLIOGRAPHY

1. Arnold, B.C., "A note on multivariate distribution with specified marginals," Journal of American Statistical Assoc. 62, 1460-1461 (1967)

2. Cramer, H., Mathematical methods of Statistics, Princeton University Press, 1946.

3. Degroot, M.H., Probability and Statistics, Addison Wesley, Menlo Park, CA (1975).

4. Fishman, G.S., Concepts and Methods in Discrete Event Digital Simulation, John Wiley and Sons, Inc., New York (1973).

5. Gaver, D.P. and Lewis, P.A.W., "First order autoregressive gamma sequences", Tech. report #NPS-55-78-016, Naval Postgraduate School, Monterey, CA (1978).

6. Gaver, D.P., "Point process problems in reliability," Stochastic Point Process (774-800), John Wiley and Sons, Inc., New York (1972).

7. Johnson, M.E., "Models and methods for generating dependent random vectors," Ph.D. dissertation, Univ. of Iowa, Iowa City, Iowa (1976).

8. Johnson, M.E. and Tenenbein, A., "Bivariate distribution with given marginals and fixed measure of dependence," Informal report LA-7700-MS, Los Alamos Scientific Laboratory, Los Alamos, New Mexico (1977).

9. Johnson, N.L. and Kotz, S., Distribution in Statistics Continuous Multivariate Distribution, John Wiley and Sons, Inc., New York (1972).

10. Kimeldorf, G. and Sampson, A., "One parameter families of bivariate distribution with fixed marginals," Communication in Statistics, 4(13), 293-301 (1975).

11. Kruskal, W.H., "Ordinal measures of association," Journal of American Statistics Assoc., 53, 814-861 (1958).

12. Lawrance, A.J. and Lewis, P.A.W., "Simulation of some autoregressive Markovian sequences of positive random variables," Tech. report NPS-55-79-024, Naval Postgraduate School, Monterey, CA (1979).

13. Mardia, K.V., Families of Bivariate Distributions, Hafner Publishing Co., Darien, Conn. (1970).

14. Marshall, A.W. and Olkin, I., "A multivariate exponential distribution," Journal of American Statistics Assoc., 62, 30-44 (1967).

15. Marshall, A.W. and Olkin, I., "A generalized bivariate exponential distribution," Journal of Applied Problems, 4, 291-302 (1967).

16. Mood, Graybill and Boes, Introduction to the Theory of Statistics, McGraw Hill, New York, 1974.

17. Moran, P.A.P., "Testing for correlation between non-negative variates," Biometrika, 54, 385-394 (1967).

18. Plackett, R.L., "A class of bivariate distributions," Journal of Amer. Statist. Assoc., 60, 516-22 (1965).

19. Ross, S.M., Introduction to Probability Models, Academic Press, New York, 1972.

20. Schmeiser, B.W. and Ram, Lal, "Computer generation of bivariate gamma random vectors," Tech. report OR-79009, Southern Methodist Univ., Dallas, Texas (1979).

21. Shannon, R.E., Systems Simulation, the Art and Science, Prentice Hall, Englewood Cliffs, New Jersey (1975).

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c.1 A general mixture-truncation method for generating bivariate random variables from univariate generators.